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Half-Space Problems for a Linearized Discrete Quantum Kinetic Equation

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Abstract We study typical half-space problems of rarefied gas dynamics, including the problems of Milne and Kramer, for a general discrete model of a quantum kinetic equation for excitations in a Bose gas. In the discrete case the plane stationary quantum kinetic equation reduces to a system of ordinary differential equations. These systems are studied close to equilibrium and are proved to have the same structure as corresponding systems for the discrete Boltzmann equation. Then a classification of well-posed half-space problems for the homogeneous, as well as the inhomogeneous, linearized discrete kinetic equation can be made. The number of additional conditions that need to be imposed for well-posedness is given by some characteristic numbers. These characteristic numbers are calculated for discrete models axially symmetric with respect to the x-axis. When the characteristic numbers change is found in the discrete as well as the continuous case. As an illustration explicit solutions are found for a small-sized model.

Keywords

of Arkeryd and Nouri [2] we are interested in the equation

$$\begin{cases} p^1 \dfrac{dF}{dx} = C_{12}(F) + G C_{22}(F), \\ F(0,\mathbf{p}) = F_0(\mathbf{p}) \text{ for } p^1 > 0, \end{cases} \tag{1}$$

where $F = F(x,\mathbf{p})$ denotes the distribution function of the excitations, $G \in \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ is constant, $x \in \mathbb{R}_+$, $\mathbf{p} = (p^1, p^2, p^3) \in \mathbb{R}^3$, and $F_0 = F_0(\mathbf{p})$ is given, with the collision integrals

$$C_{12}(F) = n \int d_0 d_3 \left((1 + F_*) F' F'' - F_* (1 + F')(1 + F'') \right) d\mathbf{p}_* \, d\mathbf{p}' d\mathbf{p}'',$$

with

$$d_0 = \delta(\mathbf{p}_* - \mathbf{p}' - \mathbf{p}'')\, \delta(|\mathbf{p}_*|^2 + n - |\mathbf{p}'|^2 - |\mathbf{p}''|^2) \quad \text{and}$$

$$d_3 = \delta(\mathbf{p} - \mathbf{p}_*) - \delta(\mathbf{p}' - \mathbf{p}) - \delta(\mathbf{p}'' - \mathbf{p});$$

and

$$C_{22}(F) = \int d_1 \left((1 + F)(1 + F_*) F' F_*' - F F_* (1 + F')(1 + F_*') \right) d\mathbf{p}_* \, d\mathbf{p}' d\mathbf{p}_*'$$

with

$$d_1 = \delta(\mathbf{p} + \mathbf{p}_* - \mathbf{p}' - \mathbf{p}_*')\, \delta(|\mathbf{p}|^2 + |\mathbf{p}_*|^2 - |\mathbf{p}'|^2 - |\mathbf{p}_*'|^2).$$

Here and below we use the notation $F' = F(x,\mathbf{p}')$ etc.. The density of the condensate, n_c, is assumed to be constant, $n_c = n$ (cf. [2]). In the Nordheim-Boltzmann [24] (or the Uehling-Uhlenbeck [28]) collision integral $C_{22}(F)$ binary collisions between excited atoms are considered, while in the collision integral $C_{12}(F)$ binary collisions involving one condensate atom are considered [33].

If the distribution function F is close to an equilibrium distribution, i.e. a Planckian

$$P = \frac{1}{e^{a(|\mathbf{p}|^2 + n) + b \cdot \mathbf{p}} - 1} = \frac{1}{e^{a(|\mathbf{p} - \mathbf{p}_0|^2 + n_0)} - 1},$$

with $a > 0$, $b \in \mathbb{R}^3$, $\mathbf{p}_0 = -\frac{b}{2}$, and $n_0 = n - |\mathbf{p}_0|^2$, cf. [2], then the non-linear equation (1) can be approximated by the linearized equation

$$\begin{cases} p^1 \dfrac{df}{dx} + Lf = 0, \; f = f(x,\mathbf{p}) \\ f(0,\mathbf{p}) = f_0(\mathbf{p}) \text{ for } p^1 > 0, \end{cases} \tag{2}$$

where

$$F = P + (P(1+P))^{1/2} f, \; F_0 = P + (P(1+P))^{1/2} f_0, \text{ and } L = L_{12} + G L_{22},$$

and

$$
\begin{aligned}
L_{22} f = \int d_1 \Big[& PP \quad P'(1+P+P\ )\ \ (P'(1+P'))^{1/2} f' + \\
& PP \quad P'(1+P+P\ )\ \ (P'(1+P'))^{1/2} f' + \\
& P(1+P'+P') \quad P'P' \ \ (P\ (1+P\ ))^{1/2} f\ \ + \\
& P\ (1+P'+P') \quad P'P' \ \ (P(1+P))^{1/2} f \Big] \, d\mathbf{p}\ \ d\mathbf{p}' d\mathbf{p}' .
\end{aligned}
$$

It can be shown (cf. [2] for L_{12} and, for example, [16] for the linearized Boltzmann operator) that the linearized operators L_{12} and L_{22}, and so also L, (all acting in the velocity space) are symmetric and positive semi-definite operators on L^2.

In the paper [2], Arkeryd and Nouri studied the Milne problem for the linearized equation (2), with $G = 0$, $F = P(1+f)$, and a cut-off at $l > 0$ in the integrand of L, such that $|\mathbf{p}|, |\mathbf{p}\ |, |\mathbf{p}'|, |\mathbf{p}'|\quad l$. The corresponding linearized half-space problems for the Boltzmann equation is well-studied [3,17,20], see also [4] and references therein.

In this paper we discretize the variable $\mathbf{p}$ and obtain a general discrete model for Eq.(1), which is similar to the discrete Boltzmann equation (a general discrete velocity model, DVM, for the Boltzmann equation) [14]. It is a well-known fact that the Boltzmann equation can be approximated up to any order of DVMs [11, 25,18], which motivated us to introduce discrete models also for this equation. By the discretization, Eq.(1) reduces to a system of ordinary differential equations. We find that the discrete linearized quantum kinetic equation (the discrete version of Eq.(2)) has the same structure as the linearized discrete Boltzmann equation. This means that the linearized operator is symmetric and positive semi-definite, and that the null-space is non-trivial. One difference is that the mass flow is not constant (with respect to the variable x) as for the discrete Boltzmann equation. However, this cause us no difficulties, in difference to in the continuous case in [2], since the structure will still be the same. A classification of well-posed half-space problems for the homogeneous, as well as the inhomogeneous, linearized discrete Boltzmann equation has been made in [5] (which is a continuation of the paper [9]), based on the dimensions of the stable, unstable and center manifolds of the singular points (Maxwellians for DVMs). We establish similar results in our case. This means, that we, in addition to adding the Nordheim-Boltzmann (or Uehling-Uhlenbeck) collision integral $C_{22}(F)$, also can introduce an inhomogeneous term and more general boundary conditions. Similar results can also be established for the discrete Nordheim-Boltzmann (or Uehling-Uhlenbeck) equation and the discrete anyon Boltzmann equation (see Remark 6 and 7 in Section 4).

Furthermore, we have, for axially symmetric discrete models with respect to the x-axis, made a table of some characteristic numbers, from which we, by Theorem 1, can obtain the dimensions of the stable, unstable and center manifolds of the singular points (Planckians in our case). This includes determining when the characteristic numbers change, not only in the discrete, but also in the continuous case (cf. [7,5] for DVMs and [17] for the Boltzmann equation).

Nonlinear half-space problems for the Boltzmann equation have also been studied for small perturbations of the singular points (Maxwellians for the Boltzmann equation), see for example [6,21,22,29] for the discrete Boltzmann equation and [30, 19, 32] for the continuous Boltzmann equation. In the discrete case similar results to the ones in [6] can be obtained for the quantum kinetic equation (1).

We want to make clear that the aim of the paper is not the study of the general half-space problem we obtain, since it is already well studied for the discrete Boltzmann equation [9, 5]. The novelty of the paper is instead the introduction of discrete models for the equation for the distribution function of excited atoms interacting with a Bose-Einstein condensate and the studies of those models. These studies includes that we by the right linearization end up with a system having similar properties as the one obtained for the discrete Boltzmann equation. It makes it, as mentioned above, possible to extend the results in [2] obtained for the continuous equation. The same is true also for the discrete Nordheim-Boltzmann (or Uehling-Uhlenbeck) equation and the discrete anyon Boltzmann equation (see Remark 6 and 7 in Section 4). However, in concrete situations, it will look different depending on which equation we study. One difference is the characteristic numbers, studied for axially symmetric models in Section 5. Our experience from the Boltzmann equation, make us believe that these numbers (also calculated in the continuous case) are as important in the continuous case as in the studied discrete case. Another difference between our equation and the Boltzmann equation, is that in our case we will have a non-constant mass-flow. To illustrate this we created a model that we solved explicitly and was able to give an explicit expression for the non-constant mass flow for (see Section 6).

The remaining part of this paper is organized as follows. In Section 2 we introduce a general discrete model for Eq.(1) and derive some of its properties. By a transformation around a Planckian, we obtain a linearized operator and a non-linear part presented in Section 3. It is shown that the system has the same structure (the linearized operator and the non-linear part have similar properties) as the corresponding system for DVMs of the Boltzmann equation. Then some results for the linearized discrete Boltzmann equation can be applied for the problem of our study. These results are presented in Section 4. In Section 5 some characteristic numbers, from which we, by Theorem 1, can obtain the dimensions of the stable, unstable and center manifolds of the singular points (Planckians), are obtained for axially symmetric discrete models with respect to the x-axis. When the characteristic numbers change, are determined both in the discrete as well as the continuous case. A linearized half-space problem (with $G = 0$) is explicitly solved for a small-sized discrete model in Section 6.

2 Discrete model

We introduce a general discrete model for Eq.(1)

$$p_i^1 \frac{dF_i}{dx} = C_{12i}(F) + G C_{22i}(F), \; x \in \mathbb{R}_+, \; i = 1, ..., N, \tag{3}$$

where $\mathscr{P} = \{\mathbf{p}_1, ..., \mathbf{p}_N\} \subset \mathbb{R}^d$ is a finite set, $F_i = F_i(x) = F(x, \mathbf{p}_i)$, where $F = F(x, \mathbf{p})$ is the distribution function of the excitations, and $G \in \mathbb{R}_+$ is constant. For generality, we allow $\mathbf{p}$ to be of dimension d, rather than of dimension 3. We assume that

$$p_i^1 \neq 0, \text{ for } i = 1, ..., N.$$

The collision operators $C_{12i}(F)$ are given by

$$C_{12i}(F) = \sum_{j;k;l=1}^{N} (d_{il} \quad d_{ij} \quad d$$

where

$$Q_i(F;G) = \frac{1}{2}\sum_{j;k;l=1}^{N}\Gamma_{ij}^{kl}\left((G_k H_l + H_k G_l) - (G_i H_j + H_j G_i)\right)$$

and

$$\hat{Q}_i(F;G;H) =$$
$$\frac{1}{2}\sum_{j;k;l=1}^{N}\Gamma_{ij}^{kl}\left((F_i + F_j)(G_k H_l + H_k G_l) - (F_k + F_l)(G_i H_j + H_j G_i)\right).$$

A function $f = f(\mathbf{p})$

or

$$P_i = \frac{M_i}{1 - M_i} = \frac{1}{e^{\alpha\left(|\mathbf{p}_i|^2+\nu\right)+\beta\,\mathbf{p}_i} - 1} \text{ for } i = 1,\ldots,N.$$

One can easily see that

$$\langle H, C_{12}(F)\rangle = \nu \sum_{i,j,k=1}^{N} G^i_{jk}\left((1+F_i)F_jF_k - F_i(1+F_j)(1+F_k)\right)(H_i - H_j - H_k), \quad (13)$$

and so

$$\log\frac{F}{1+F}, C_{12}(F) = \nu \sum_{i,j,k=1}^{N} G^i_{jk}(1+F_i)(1+F_j)(1+F_k)$$

for all indices such that $G_{ij}^{kl} \neq 0$

where

$$L^i_{jk} f = (1+P_j+P_k) R_i^{1/2} f_i - (P_k - P_i) R_j^{1/2} f_j - (P_j - P_i) R_k^{1/2} f_k;$$

and

$$S_{12i}(f,g) = n \sum_{j,k=1}^{N} \frac{\Gamma^i_{jk} S^i_{jk}(f,g) - 2\Gamma^k_{ij} S^k_{ij}(f,g)}{R_i^{1/2}}, \quad i = 1,\ldots,N,$$

with

$$S^i_{jk}(f,g) = \frac{1}{2} \Big(R_j^{1/2} R_k^{1/2} (f_j g_k + g_j f_k) - R_i^{1/2} R_j^{1/2} (f_i g_j + g_i f_j) - R_i^{1/2} R_k^{1/2} (f_i g_k + g_i f_k) \Big).$$

Moreover, the operators (22) and (24) read, in more explicit forms,

$$(L_{22} f)_i = \sum_{j,k,l=1}^{N} \frac{\Gamma^{kl}_{ij}}{R_i^{1/2}} \left(P^{kl}_{ij} f_i + P^{kl}_{ji} f_j - P^{ij}_{kl} f_k - P^{ij}_{lk} f_l \right), \quad i = 1,\ldots,N \tag{26}$$

where

$$P^{kl}_{ij} = \left(P_j (1+P_k+P_l) - P_k P_l \right) R_i^{1/2},$$

and

$$S_{22i}(f,f,f) = \sum_{j,k,l=1}^{N} \frac{\Gamma^{kl}_{ij}}{R_i^{1/2}} \left(S^{kl}_{ij}(f,f,f) - S^{ij}_{kl}(f,f,f) \right), \quad i = 1,\ldots,N,$$

with

$$S^{kl}_{ij}(f,f,f) = (1+P_i+P_j) R_k^{1/2} R_l^{1/2} f_k f_l + \left(R_i^{1/2} f_i + R_j^{1/2} f_j \right) \Big(P_k R_l^{1/2} f_l + P_l R_k^{1/2} f_k + R_k^{1/2} R_l^{1/2} f_k f_l \Big).$$

By Eqs.(4),(25), and the relations

$$P_j(1+P_j)(P_k - P_i) = P_k(1+P_k)(P_j - P_i) = P_i(1+P_j)(1+P_k);$$
$$P_i(1+P_j+P_k) = P_2 P_3 = P_i(1+P_j)(1+P_k)$$

for $\Gamma^i_{jk} \neq 0$, we obtain the equality

$$\langle g, L_{12} f \rangle = n \sum_{i,j,k=1}^{N} \Gamma^i_{jk} P_i (1+P_j)(1+P_k) \left(\frac{f_i}{R_i^{1/2}} - \frac{f_j}{R_j^{1/2}} \right.$$

Similarly, by Eqs.(5),(26), and the relations

$$P_i P_j (1+P_k)(1+P_l) = P_k P_l (1+P_i)(1+P_j);$$
$$P_{ij}^{kl} = P_k P_l (1+P_j) \frac{\sqrt{1+P_i}}{\sqrt{P_i}}$$

for $G_{ij}^{kl} \neq 0$, we obtain the equality

$$\langle g, L_{22} f \rangle = \frac{1}{4} \sum_{i,j,k,l=1}^{N} G_{ij}^{kl} P_i P_j (1+P_k$$

The diagonal matrix B (6) (under our assumptions) has no zero diagonal elements and is non-singular. We denote $f(0) = f_0$. Then the formal solution of Eq.(29) reads

$$f(x) = e^{-xB^{-1}L} f_0 + \int_0^x e^{(s-x}$$

4 Half-space problems

We consider the inhomogeneous (or homogeneous if $g = 0$) linearized problem

$$B\frac{df}{dx} + Lf = g, \tag{33}$$

where $g = g(x) \in L^1(\mathbb{R}_+; \mathbb{R}^n)$, with one of the boundary conditions

(O) the solution tends to zero at infinity, i.e.

$$f(x) \to 0 \text{ as } x \to \infty;$$

(P) the solution is bounded, i.e.

$$|f(x)| < \infty \text{ for all } x \in \mathbb{R}_+;$$

(Q) the solution can be slowly increasing at infinity, i.e.

$$|f(x)|e^{-\varepsilon x} \to 0 \text{ as } x \to \infty, \text{ for all } \varepsilon > 0.$$

In case of boundary condition (O) we additionally assume that

$$g(x) \in N(L)^{\perp} \text{ for all } x \in \mathbb{R}_+. \tag{34}$$

Remark 2 The boundary condition (O) corresponds to the case when we have made the expansion (20) around a Planckian P, such that $F \to P$ as $x \to \infty$. The boundary conditions (P) and (Q) are the boundary conditions in the Milne and Kramers problem respectively.

We can (without loss of generality) assume that

$$B = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}, \tag{35}$$

where

$$B_+ = \operatorname{diag}\left(p^1_1, \dots, p^1_{n^+}\right) \text{ and } B_- = \operatorname{diag}\left(p^1_{n^+}\right.$$

where (in the notations of (30)-(32))

$$Y^+(x) = \sum_{r=1}^{m^+} u_r \left(b_r(0) e^{-\lambda_r x} + \int_0^x e^{(s-x)\lambda_r} \frac{\langle g(s); u_r \rangle}{\lambda_r} ds \right),$$

$$Y^-(x) = \sum_{r=m^+ +1}^{q} u_r \int_x^{\infty} e^{(s-x)\lambda_r} \frac{\langle g(s); u_r \rangle}{\lambda_r} ds,$$

$$F^+(x) = \sum_{i=1}^{k^+} y_i \left(m_i(0) + \int_0^x \frac{\langle g(s); y_i \rangle}{h} \right.$$

is a basis of R^{n^+} (see [9,5,6]), we obtain the following theorem (cf. [5]), where we denote

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Remark 3 We can also, before prescribing the set of velocities, make the change of variables $\mathbf{p} \to \mathbf{p} + \mathbf{p}_0$ (cf. Eq.(19)). We then, instead of relations (4), obtain the relations

$$\mathbf{p}_i = \mathbf{p}_j + \mathbf{p}_k + \mathbf{p}_0 \text{ and } |\mathbf{p}_i|^2 = \left|\mathbf{p}_j\right|^2 + |\mathbf{p}_k|^2 + n_0,$$

and the collision invariants

$$f = \mathbf{a} \cdot (\mathbf{p} + \mathbf{p}_0) + b\left(|\mathbf{p}|^2 + n_0\right).$$

Moreover

$$N(L) = \operatorname{span}\left\{R^{1/2}\left(p^1 + p_0^1\right), \dots, R^{1/2}\left(p^d + p_0^d\right), R^{1/2}\left(|\mathbf{p}|^2 + n_0\right)\right\};$$

and if $p_0^1 \neq 0$, then the matrix B have to be replaced with $B + p_0^1 I$.

Remark 4 Our results can be generalized to more general boundary conditions

$$f^+(0) = Cf^-(0) + h_0,$$

where C is a given $n^+ \times n^-$ matrix and $h_0 \in \mathbb{R}^{n^+}$ (cf. [5,6]).

In order to be able to obtain existence and uniqueness of solutions of the linearized half-space problems we will then need to assume that the matrix C fulfills the condition

$$\dim(R_+ - CR_-)U_+ = m^+,$$

with $U_+ = \operatorname{span}(u_1, \dots, u_{m^+})$, as we consider boundary condition (O), the condition

$$\dim(R_+ - CR_-)X_+ = n^+, \tag{39}$$

with $X_+ = \operatorname{span}(u_1, \dots, u_{m^+}, y_1, \dots, y_{k^+}, z_1, \dots, z_l)$, as we consider boundary condition (P), and the condition (39) or the condition

$$\dim(R_+ - CR_-)\tilde{X}_+ = n^+,$$

with $\tilde{X}_+$

with

where it is assumed that the collision coefficients Γ_{ij}^{kl} satisfy the relations

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0,$$

with equality unless the conservation laws (5) are satisfied.

Here $\epsilon = 0$ corresponds to the discrete Boltzmann equation, and we have $\epsilon = 1$ for bosons and $\epsilon = -1$ for fermions.

The singular points are

$$P = \frac{M}{1 - \epsilon M},$$

where $M = e^{a+\mathbf{b}\cdot\mathbf{p}+c|\mathbf{p}|^2}$, with $a, c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$, is a Maxwellian (note that for $\epsilon = 0$, $P = M$).

By denoting (cf. Eq.(20))

$$f = P + \sqrt{R}F, \text{ with } R = P(1+\epsilon P),$$

we obtain a system, where the linearized operator L is symmetric and positive semi-definite, with a non-trivial null-space $N(L)$, which for normal models will be

$$N(L) = \text{span}\left(R^{1/2}, R^{1/2}p^1, \ldots, R^{1/2}p^d, R^{1/2}|\mathbf{p}|^2\right).$$

For $\epsilon = 0$ this is a well-known fact, see for example [5], and for $\epsilon = 1$, $L = L_{22}$, where L_{22} is given in Eq.(22), in the particular case $\Gamma_{ij}^{kl} = 1$ for all non-zero Γ_{ij}^{kl}.

Remark 7 We introduce a discrete version of the anyon Boltzmann equation [8, 1]:

$$p_i^1 \frac{dg_i}{dx} = Q_i^\alpha(g), \; x \in \mathbb{R}_+, \; i = 1, \ldots, N,$$

where $0 \leq \alpha \leq 1$ ($\alpha = 0$ for bosons and $\alpha = 1$ for fermions) and

$$Q_i^\alpha(g) = \sum_{j,k,l=1}^{N} \Gamma_{ij}^{kl} \left(g_k g_l F(g_i) F(g_j) - g_i g_j F(g_k) F(g_l)\right),$$

with

$$F(h) = (1 - \alpha h)^\alpha (1 + (1-\alpha) h)^{1-\alpha},$$

where it is assumed that the collision coefficients Γ_{ij}^{kl} satisfy the relations

$$\Gamma_{ij}^{kl} = \Gamma_{ji}^{kl} = \Gamma_{kl}^{ij} \geq 0,$$

with equality unless the conservation laws (5) are satisfied.

The singular points g_0 are given by

$$\frac{g_0}{F(g_0)} = M$$

where $M = e^{a+\mathbf{b}\cdot\mathbf{p}+c|\mathbf{p}|^2}$, with $a, c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$, is a Maxwellian, or (cf. [31])

$$g_0 = \frac{1}{w(\mathbf{p}) + \alpha},$$

where

$$w(\mathbf{p})^{a}\,(1 + w(\mathbf{p}))^{1\ \ a} =$$

Here we study instead of Eq.(33), cf. Remark 3, the equation

$$(B + p_0^1 I)\frac{df}{dx} + Lf = g.$$

The linearized collision operator L has the null-space

$$N(L) = \mathrm{span}\,(f_1; f_2)\,,$$

where

$$f_1 = R^{1=2}\;\; p^1 + p_0^1 \;\; = R^{1=2}\;\; (p_1^1 + p_0^1; :::; p_N^1 + p_0^1; \;\;\; p_1^1 + p_0^1; :::; \;\;\; p_N^1 + p_0^1)$$
$$f_2 = R^{1=2} j\mathbf{p}j^2 = R^{1=2}\;\; (j\mathbf{p}_1 j^2 ; :::; j\mathbf{p}_N j^2 ; j\mathbf{p}_1 j^2 ; :::; j\mathbf{p}_N j^2);$$

Similar numbers have been calculated for axially symmetric DVMs around the x-axis in [7, 5].

For the continuous Boltzmann equation the degenerate values are 0 and $\sqrt{\frac{5T}{3}}$, where T is the temperature of the Maxwellian M, that the linearization is made around (cf. Eq.(20)) [17]. The values of k^+, k and l for the Boltzmann equation are given by the table (cf. [17])

		$u = \sqrt{\frac{5T}{3}}$		$u = 0$		$u = \sqrt{\frac{5T}{3}}$

We want to stress that $I_2 I_6$ I_4^2 by the Cauchy-Schwarz inequality. Furthermore, the values of k^+, k and l are given by the table

		$p_0^1 = \quad p_{0+}^1$		$p_0^1 = 0$		$p_0^1 = p_{0+}^1$	
k^+	0	0	1	1	3	3	4
k	4	3	3	1	1	0	0
l	0	1	0	2	0	1	0

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For a flow axially symmetric around the x-axis we can reduce the system (42) to

$$\begin{cases} 2p\dfrac{dF_1}{dx} = 2n\left((1+F_1)F_2^2 - F_1(1+F_2)^2\right) \\ 2p\dfrac{dF_2}{dx} = -4n\left((1+F_1)F_2^2 - F_1(1+F_2)^2\right) \\ \quad 2p\dfrac{dF_3}{dx} = 2n\left((1+F_3)F_4^2 - F_3(1+F_4)^2\right) \\ \quad 2p\dfrac{dF_4}{dx} = -4n\left((1+F_3)F_4^2 - F_3(1+F_4)^2\right) \end{cases} . \tag{43}$$

with $F_1 = \tilde{F}_1$, $F_2 = \tilde{F}_2 = \tilde{F}_3$, $F_3 = \tilde{F}_4$, and $F_4 = \tilde{F}_5 = \tilde{F}_6$. Note that the collision invariants are p^1 and $|\mathbf{p}|^2$.

We define the projections $R_+ : \mathbb{R}^4 \to \mathbb{R}^2$ and $R_- : \mathbb{R}^4 \to \mathbb{R}^2$, by

$$R_+h = h^+ = (h_1, h_2) \text{ and } R_-h = h^- = (h_3, h_4),$$

where $h = (h_1, h_2, h_3, h_4)$.

We consider the problem

$$\begin{cases} B\dfrac{dF}{dx} = C_{12}(F), \\ F^+(0) = \tilde{h}_0 \end{cases} ,$$

where

$$B = (2p, 2p, -2p, -2p)$$

and

$$C_{12}(F) = 2n\left((1+F_1)F_2^2 - F_1(1+F_2)^2\right)(1, -2, 0, 0) + \\ 2n\left((1+F_3)F_4^2 - F_3(1+F_4)^2\right)(0, 0, 1, -2).$$

If we denote

$$F = P + R^{1/2} f,$$

with

$$R = P(1+P) \text{ and } P = \left(\frac{1}{e^{2p^2}-1}, \frac{1}{e^{2p^2}+1}; 1; \frac{1}{e^{2p^2}+1}; 1\right),$$

in Eq.(43) we obtain

$$\begin{cases} B\dfrac{df}{dx} + Lf = S(f) \\ f^+(0) = h_0 \end{cases} ,$$

where L is the linearized collision operator, $S(f)$ the nonlinear part, and $h_0 \in \mathbb{R}^2$.
The linearized problem reads

$$\begin{cases} B\dfrac{df}{dx} + Lf = 0 \\ f^+(0) = h_0 \end{cases} . \tag{44}$$

where the linearized collision operator

$$L = \frac{2n}{\sinh(p^2)} \begin{pmatrix} \cosh(p^2) & 1 & \\ 1 & \frac{1}{\cosh(p^2)} & \\ 0 & 0 & \dots \end{pmatrix}$$

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is symmetric and positive semi-de...

$N(L) = \mathrm{span}(R^{1/2}p^1; R^{1/2}|\mathbf{p}|^2) = \dots ; \; 0; 0; 1; \cosh(p^2) \;).$

Since

$K = \langle y_1; y_1 \rangle_B \; \langle y_1; y_2 \rangle_B \; 1 \;).$

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