Weak Shock Wave Solutions for the Discrete Boltzmann Equation

N. Bernhoff and A. V. Bobylev

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Abstract. The analytically difficult problem of existence of shock wave solutions is studied for the general discrete velocity model (DVM) with an arbitrary finite number of velocities (the discrete Boltzmann equation in terminology of H. Cabannes). For the shock wave problem the discrete Boltzmann equation becomes a system of ordinary differential equations (dynamical system). Then the shock waves can be seen as heteroclinic orbits connecting two singular points (Maxwellians). In this work we give a constructive proof for the existence of solutions in the case of weak shocks.

We assume that a given Maxwellian is approached at infinity, and consider shock speeds close to a typical speed c_0, corresponding to the sound speed in the continuous case. The existence of a non-negative locally unique (up to a shift in the independent variable) bounded solution is proved by using contraction mapping arguments (after a suitable decomposition of the system). This solution is then shown to tend to a Maxwellian at minus infinity.

Existence of weak shock wave solutions for DVMs was proved by Bose, Illner and Ukai in 1998. In their technical proof Bose et al. are following the lines of the pioneering work for the continuous Boltzmann equation by Caflisch and Nicolaenko. In this work, we follow a more straightforward way, suiting the discrete case. Our approach is based on results by the authors on the main characteristics (dimensions of corresponding stable, unstable and center manifolds) for singular points to general dynamical systems of the same type as in the shock wave problem for DVMs. Our proof is constructive, and it is also shown (at least implicitly) how close to the typical speed c_0, the shock speed must be for our results to be valid. All results are mathematically rigorous.

Our results are also applicable for DVMs for mixtures.

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INTRODUCTION

We are concerned with the existence of shock wave solutions $f = f(\ ^1, \xi, t) = \ (\ ^1 - ct, \xi)$, of the Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_{\mathbf{x}} f = \ (f, f).$$

Here $\mathbf{x} = (\ ^1, \dots, \ ^d) \in \mathbb{R}^d$, $\xi = (\xi^1, \dots, \xi^d) \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$ denote position, velocity and time respectively. Furthermore, $c > c_0$ denotes the speed of the wave, where c_0 is the speed of sound. The solutions are assumed to approach two given Maxwellians $M_\pm = \frac{\rho_\pm}{(2\pi\ _\pm)^{d/2}} e^{-|\xi - \mathbf{u}_\pm|^2/(2\ _\pm)}$ (ρ, $\mathbf{u}$ and denote density, bulk velocity and temperature respectively) as $\ \to \pm\infty$, which are related through the Rankine-Hugoniot conditions.

The (shock wave) problem is to find a solution $\ = \ (y, \xi)$ ($y = \ ^1 - ct$) of the equation

$$(\xi^1 - c)\frac{\partial}{\partial y} = \ (\ ,\ ), \tag{1}$$

such that

$$f \to M_\pm \text{ as } y \to \pm\infty. \tag{2}$$

In this paper, we consider the shock wave problem (1), (2) for the general discrete velocity model (DVM) (the discrete Boltzmann equation) [1, 2]. We allow the velocity variable to take values only from a finite subset V of $\mathbb{R}^d$, i.e. $\xi \in \mathbf{V} = \{\xi_1, \dots, \xi_n\} \subset \mathbb{R}^d$, where n is an arbitrary natural number.

We obtain, from Eq.(1), a system of ODEs (dynamical system)

$$(\xi_i^1 - c)\frac{d\ _i}{dy} = \ _i(\ ,\ ),\ i = 1, \dots, n,\ c \in \mathbb{R},$$

where $= (_1, \ldots, _n)$, with $= (y) = (y, \xi), = 1, \ldots, n$. The collision operator $= (_1, \ldots, _n)$ is given by

$$(,) = 1$$

2. The vector(s) $\varphi_\square$ fulfilling Eqs.(4), also satisfy $\langle M_+\varphi_\square, \varphi_\square^2\rangle_{-c_0} \neq 0$. We choose the sign of the vector $\varphi_\square$, such that $\langle M_+\varphi_\square, \varphi_\square^2\rangle_{-c_0} > 0$.
3. For each c (at least in a neighborhood of c_0), the number of Maxwellians M, such that the relations

$$\langle M, \varphi\rangle_{-c} = \langle M_+, \varphi\rangle_{-c}, \quad = 1, \ldots, \qquad (5)$$

are fulfilled, is finite.

Remark 1 *Let M_+ be a Ma e an e b ve c y* $\mathbf{u} = 0$ *en f t e c nt t ann e at n* $M_+ = \frac{\rho}{(2\pi\ )^{d/2}} e^{-|\xi|^2/(2\ )}$ *n t ca e d* $= 3$ $c_0 = \pm\sqrt{\frac{5}{3}}$ *n te t at t e a t n 1 neve f ed n t e c nt n ca e* $\varphi_\square = \sqrt{\frac{1}{2\rho}}(\xi^1 \pm \sqrt{\frac{|\xi|^2}{15}})$ *and* $\langle M_+\varphi_\square, \varphi_\square^2\rangle_{-c_0} = \frac{2}{3}\sqrt{\frac{2}{\rho}} > 0$ *A t n 3 a f ed n t e c nt n ca e e e at t ne e Ma e an M be de M_+ f* (5)

Remark 2 *A e t at e ave an a y y et n a de* $(\xi^1, \ldots, \xi^d) \in \mathbf{V}$ *t en* $(\pm\xi^1, \ldots, \pm\xi^d) \in \mathbf{V}$ *n* $= 2$ *Let* $M = e^{c|\xi|^2}$ *and den te*

$$\begin{cases} \varphi_1 = (1, \ldots, 1) \\ \varphi_2 = (\xi_1^1, \ldots, \end{cases}$$

BRIEF PRESENTATION OF THE PROOF

We consider

$$(\quad - c \ ) \frac{d}{dy} = \quad (\ , \quad) \text{, where } \quad \rightarrow M_{+} \text{ as } y \rightarrow \infty,$$

and denote

$$= M + M^{1/2} \quad \text{, with } M = M_{+}.$$

We obtain

$$(\quad - c \ ) \frac{d}{dy} + L \quad = \ (\ , \ ) \text{, where } \quad \rightarrow 0 \text{ as } y \rightarrow \infty, \tag{6}$$

with

$$L \quad = -2M^{-1/2} \quad (M, M^{1/2} \quad) \text{ and } \quad , \quad) = M^{-1/2} \quad (M^{1/2} \quad , M^{1/2}$$

We denote

$$=\sum_{=0} \quad \text{where} \quad = (y) = \ , \quad -c \ .$$

Then,

$$\frac{d}{dy} + \lambda = (X,X), \text{ where } X = ({}_0, \dots,) \ = (X,X) = \sum_{j, =0} j \ \ j, \ = 0, \dots, \ , \text{ with}$$

$$j = \frac{1}{\lambda} \langle \ (j,) \rangle = \left\langle L^{-1/2} \ ((-c)^{-1} L^{1/2} \ j, (-c)^{-1} L^{1/2} \ ) \right\rangle .$$

We denote by $\widehat{\ }$ the symmetric $(+1) \times (+1)$ matrix with entries

$$(\widehat{\ })_j = j \ , \ 0 \leq j, \ \leq \ ,$$

and by $\mathscr{G} > 0$ the maximum of the absolute values of the eigenvalues of the matrix $\widehat{\ }$, or, equivalently, $\mathscr{G} = \sup_{|X|=1} |\widehat{\ } X|$.

Then

$$(X$$

Furthermore, if the functions $= (t)$,