Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org
pp. **X–XX**

ON HALF-SPACE PROBLEMS FOR THE WEAKLY NON-LINEAR DISCRETE BOLTZMANN EQUATION

Niclas Bernhoff

Department of Mathematics
Karlstad University
651 88 Karlstad, Sweden

(Communicated by the associate editor name)

Abstract. Existence of solutions of weakly non-linear half-space problems for the general discrete velocity (with arbitrarily finite number of velocities) model of the Boltzmann equation are studied. The solutions are assumed to tend tow

[17], with inflow boundary condition reads

$$\xi^1 \frac{\partial F}{\partial x} = Q(F,F), \quad F = F(x; \ldots$$

-.1 9.0Fdbin861 98oun31;9.96263.6.16 0 Td [(F)]TJ/F8

$$\begin{cases} \left(\xi^1 + u^1\right) \dfrac{\partial f}{\partial x} + L_0 f = S_0(f, f) \\ f(0, \xi) = f_0(\xi) \text{ for } \xi^1 + u^1 > 0, \\ f \to 0 \text{ as } x \to \infty, \end{cases} \tag{3}$$

where $L_0 f = -2M_0^{-1/2} Q\left(M_0, M_0^{1/2} f\right)$, $S_0(f, f) = M_0^{-1/2} Q\left(M_0^{1/2} f, M_0^{1/2} f\right)$ and $M_0 = \frac{1}{(2\pi T_\infty)^{3/2}} e^{-|\xi|^2/(2T_\infty)}$ (cf. Ref. [21]).

The general boundary condition at $x = 0$ (at the wall) in Eq. (1) reads:

$$F(0, \xi) = g_0(\xi) + \int_{\xi_*^1 < 0} K(\xi, \xi_*) F(0, \xi_*) \, d\xi_* \quad \text{for } \xi^1 > 0, \tag{4}$$

where (i) $g_0(\xi) \geq 0$ for $\xi^1 > 0$; (ii) the kernel $K(\xi, \xi_*)$, fulfills $K(\xi, \xi_*) \geq 0$ for $\xi^1 >$

by DVMs, see Refs. [12], [22], [31] and [32], and that these approximations can be used for numerical methods. The study of DVMs can also give a better conceptual understanding and new ideas, which can be applied to the Boltzmann equation. In the planar stationary case, the general DVM reduces to a system of ordinary differential equations. We continue here the study of DVMs in the directions formulated in Refs. [9], [10] and [7]. Important tools in these studies are the results in Ref. [10] (see Section 2.1 below) on the dimensions of the stable, unstable and center manifolds of the singular points (Maxwellians for DVMs).

Half-space problems for the Boltzmann equation are of great importance in the study of the asymptotic behavior of the solutions of boundary value problems of the Boltzmann equation for small Knudsen numbers, see Refs. [16] and [17]. For a comprehensive and detailed description of the asymptotic theory see Refs. [34] and [35]. The half-space problems provide the boundary conditions for the fluid-dynamic-type equations and Knudsen-layer corrections to the solution of the fluid-dynamic-type equations in a neighborhood of the boundary. Mathematical results on the half-space problem for the Boltzmann equation for a single-component gas are reviewed in Ref. [6]. Sone and Aoki with coworkers have under a long time considered problems related to these questions, both from a theoretical and numerical point of view, see Refs. [34] and [35] and references therein.

The half-space problems for the linearized Boltzmann equation are well investigated, see Refs. [5], [21] and [27]. A classification of well-posed half-space problems for the homogeneous, as well as the inhomogeneous, linearized discrete Boltzmann equation has been made in Ref. [7], based on results obtained in Ref. [10]. The results in Ref. [5 the singular point (Maxwellian for DVMs) approached at in nity is xed and small deviations of the solutions from the singular point is studied. The data for the outgoing particles at the boundary are assigned, possibly linearly depending on the data for the incoming particles. The conditions on the data at the boundary needed for the existence of a unique (in a neighborhood of the assigned Maxwellian) solution of the problem are investigated. In the non-degenerate case (corresponding, in the continuous case, to the case when the Mach number at in nity is di erent of -1, 0 and 1) implicit conditions have been found by using arguments by Ukai, Yang and Yu in Ref. [38] for the continuous Boltzmann equation. Furthermore, under certain assumptions explicit conditions are found, both in the non-degenerate and degenerate cases. The results extend, not only by more general boundary conditions, but also by more general assumptions, previous results for the discrete Boltzmann equation by Ukai in Ref. [37], and Kawashima and Nishibata in Refs. [28] and [29], and include also (for DVMs) the results obtained by Ukai, Yang and Yu in Ref. [38] for the continuous Boltzmann equation. Applications to axially symmetric models have also been studied, generalizing the results by Babovsky in Ref. [2].

All results are obtained for an arbitrary nite number of velocities. Similar results as in this paper can also be obtained for DVMs for mixtures. Existence of weak shock wave solutions for the discrete Boltzmann equation has also been proved based on the same ideas in Ref. [8].

This paper is organized as follows: In Section 2, we introduce the planar stationary discrete Boltzmann equation and review some of its properties. We make an expansion around an equilibrium Maxwellian, and review, Theorem 2.1 in Subsection 2.1, the results in Ref. [10] on the dimensions of the stable, unstable and center manifolds of the system of ODEs. The problem and the main results on existence and uniqueness are stated in Section 3 (Theorem 3.1 and Theorem 3.2). The boundary conditions at the "wall" are discussed in more detail in Section 4. In particular, in ow boundary conditions and Maxwell-type boundary conditions (Subsection 4.1) are considered. The results of [10] (stated in Theorem 2.1) are used to investigate the number of additional conditions needed to obtain well-posedness of the weakly non-linear problem in Section 5 and Section 6 respectively, and thereby to prove Theorem 3.1 (Section 5) and Theorem 3.2 (Section 6) in Section 3. Implicit conditions for the existence of a unique (in a neighborhood of the assigned Maxwellian) solution in the non-degenerate case and also for the degenerate case, but then with some restrictions on the non-linear part of the collision operator, are obtained (Section 5). The results are in accordance with corresponding results for the continuous Boltzmann equation obtained in the non-degenerate case, with in ow boundary conditions in Ref. [38, 6 The2eint3229(explicit2e)(in)289(an)]Trcit2e on

Perthame and Sulem in Ref. [26] for the Boltzmann equation, and by Babovsky in Ref. [2] for DVMs (with quite restrictive conditions on the non-linear part of the collision operator). We first consider a plane 12-velocity DVM in Subsection 8.2, but also a more general axially symmetric DVM (cf. Ref. [2]) in Subsection 8.3.

2. **Discrete Boltzmann equation.** The planar stationary system for the discrete Boltzmann equation (DBE) reads

$$\xi_i^1 \frac{dF_i}{dx} = Q_i(F,F),\ x \in \mathbb{R}_+,\ i = 1,\dots,n, \tag{5}$$

where $\mathcal{V} = \{\xi_1,\dots,\xi_n\}$, $\xi_i \in \mathbb{R}^d$

for all non-negative functions F. We have the trivial collision invariants (also called the physical collision invariants) $_0 = 1$, $_1 = {}^1; \dots; {}_d = {}^d$, $_{d+1} = |\ |^2$ (including all linear combinations of these). Here and below, we denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product on $\mathbb{R}^n$.

We consider below (even if this restriction is not necessary in our general context) only normal DVMs. That is, DVMs without spurious (or non-physical) collision invariants, i.e. any collision invariant is of the form

$$= a + \mathbf{b} \quad + c|\ |^2 \tag{11}$$

for some constant $a, c \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^d$ (methods of their construction are described in Refs. [11], [13] and [14]). In this case the equation (10) has the general solution (11).

A Maxwellian distribution (or just a Maxwellian) is a function $M = M(\ )$, such that

$$Q(M, M) = 0 \text{ and } M > 0.$$

All Maxwellian distributions are of the form

$$M = e \quad = Ae^{\mathbf{b} \quad + c|\ |^2}, \text{ with } A = e^a > 0 \text{ and } c < 0, \tag{12}$$

where is a collision invariant (11) (the latter equality is due to the assumption of normal DVMs). In general a, $\mathbf{b}$ and c can be functions of x, but since we assume that our solutions tend to a global, i.e. with absolute constant a, $\mathbf{b}$ and c, Maxwellian at infinity, our interest is in global Maxwellians, and so when we below refer to a Maxwellian, we will mean a global Maxwellian.

Given a Maxwellian M we denote

$$F = M + M^{1/2} f, \tag{13}$$

in Eq. (5), and obtain

$_i^1$ df whet1 0 T/F1lTq50

Then the system (6) transforms into

$$B\frac{df}{dx} + Lf = S(f; f). \tag{16}$$

The diagonal matrix

The Jordan normal form of $B^{-1}L$ (with respect to the basis (17)-(19)) is

$$\begin{pmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_q & & & & & & \\ & & & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & & & \\ & & & & & & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} & & \\ & & & & & & & \ddots & \\ & & & & & & & & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \end{pmatrix},$$

where there are l blocks of the type $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. For any $h \in \mathbb{R}^n$, we obtain

$$e^{-xB^{-1}L}h = \sum_{i=1}^{k} \alpha_i y_i + \sum_{j=1}^{l} ((\beta_j - x\gamma_j) z_j + \gamma_j w_j) + \sum_{r=1}^{q} \delta_r e^{-\lambda_r x} u_r,$$

where

$$\alpha_i = \frac{\langle h, y_i \rangle_B}{\langle y_i, y_i \rangle_B}, \quad \delta_r = \frac{\langle h, u_r \rangle_B}{\lambda_r}, \quad \beta_j = \langle h, z_j \rangle_B \text{ and } \gamma_j = \langle h, w_j \rangle_B .$$

3. Statement of the problem and main results.

We consider the non-linear system

$$B\frac{df}{dx} + Lf = S(f, f), \tag{20}$$

where the solution tends to zero at infinity, i.e.

$$f(x) \to 0 \text{ as } x \to \infty, \tag{21}$$

and

$$S(f, f) \in N(L)^{\perp}. \tag{22}$$

The boundary conditions (21) correspond to the case when we have made the expansion (13) around a Maxwellian $M = M_\infty$, such that $F \to M_\infty$ as $x \to \infty$.

We can (without loss of generality) assume that

$$B = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix},$$

where

$$B_+ = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n_+}) \text{ and } B_- = \operatorname{diag}(\lambda_{n_++1}, \ldots, \lambda_n),$$

$$\text{with } \lambda_1, \ldots, \lambda_{n_+} > 0 \text{ and } \lambda_{n_++1}, \ldots, \lambda_n < 0. \tag{23}$$

We also define the projections $R_+ : \mathbb{R}^n \to \mathbb{R}^{n_+}$ and $R_- : \mathbb{R}^n \to \mathbb{R}^{n_-}$, $n_- = n - n_+$, by B

where C is a given $n^+ \times n^-$ matrix and $h_0 \in \mathbb{R}^{n^+}$. We introduce the operator $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^{n^+}$, given by

$$\mathcal{C} = R_+ - CR_- .$$

We will also assume that the matrix C fulfills one of the conditions

$$\dim \mathcal{C}U_+ = m^+,$$
$$\text{with } U_+ = \operatorname{span}(u \mid Lu = \lambda Bu, \text{ with } \lambda > 0) = \operatorname{span}(u_1, \dots, u_{m^+}), \tag{25}$$

and

$$\dim \mathcal{C}X_+ = n^+, \text{ with } X_+ = \operatorname{span}(u_1, \dots, u_{m^+}, y_1, \dots, y_{k^+}, z_1, \dots, z_l). \tag{26}$$

Remark 3. For the continuous Boltzmann equation (with $d = 3$), if we have made the expansion (13) around a non-drifting Maxwellian

$$M = \frac{1}{(2\pi T_\infty)^{3/2}} e^{-|\xi|^2/2T_\infty},$$

$k^+ = 1$, $l = 3$ and the collision invariants y_1, y_2, z_1, z_2 and z_3 can be chosen as, cf. Ref. [21],

$$y_1 = \left(-\frac{\xi_1}{\sqrt{2T_\infty}} + \frac{|\xi|^2}{\sqrt{30T_\infty}} \right) M^{1/2}, \quad y_2 = \left(\frac{\xi_1}{\sqrt{2T_\infty}} + \frac{|\xi|^2}{\sqrt{30T_\infty}} \right) M^{1/2},$$

$$z_1 = \left(\sqrt{\frac{5}{2}} - \frac{|\xi|^2}{\sqrt{10T_\infty}} \right) M^{1/2}, \quad z_2 = \frac{\xi_2}{\sqrt{T_\infty}} M^{1/2} \text{ and } z_3 = \frac{\xi_3}{\sqrt{T_\infty}} M^{1/2}.$$

Moreover,

$$w_j = L^{-1}\xi_1 z_j,$$

in the continuous case, and the continuous analogue of equation $Lu = \lambda Bu$ is

$$Lh = \lambda^{-1}\xi_1 h, \; h = h(\xi), \tag{27}$$

(see Ref. [16] for a discussion on the eigenvalue problem (27)). We also want to point out that, in the continuous case, the boundary conditions (before the expansion (13)), that correspond to conditions (24), are given by Eqs. (4).

We now state our main results.

Theorem 3.1. *Let condition (26) be fulfilled and suppose that* $\langle S(f(x), f(x)), w_j \rangle = 0$ *for* $j = 1, \dots, l$, *and that* $\langle h_0, h_0 \rangle_{B_+}$ *is sufficiently small. Then with* $k^+ + l$ *conditions on* h_0*, the system (20) with the boundary conditions (21) and (24) has a locally unique solution.*

Theorem 3.1 is proved in Section 5.

Theorem 3.2. *Let condition (25) be fulfilled and assume that*

$$h_0, \; \mathcal{C}e^{xB^{-1}L}B^{-1}S(f(x), f(x)) \in \mathcal{C}U_+ \text{ for all } x \in \mathbb{R}_+,$$
$$\text{with } U_+ = \operatorname{span}(u : Lu = \lambda Bu, \; \lambda > 0) = \operatorname{span}(u_1, \dots, u_{m^+}). \tag{28}$$

Then there is a positive number δ_0*, such that if*

$$|h_0| \le \delta_0,$$

then the system (20) with the boundary conditions (21) and (24) has a locally unique solution.

The proof of Theorem 3.2 is outlined in Section 6.

Remark 4. If condition (25) is fulfilled, then the condition

$$h_0 \in CU_+.$$

implies that we have $k^+ + l$ conditions on h_0.

Remark 5. If the conditions

$$\begin{aligned} CU_- &\subseteq CU_+, \\ \text{with } U_- &= \operatorname{span}\left(\{u \mid Lu = \lambda Bu, \text{ with } \lambda < 0\} \cup \{z_1, ..., z_l\}\right) \\ &= \operatorname{span}\left(u_{m^+ + 1}, ..., u_q, z_1, ..., z_l\right); \end{aligned} \tag{29}$$

and (22) are fulfilled, then

$$Ce^{xB^{-1}L}B^{-1}S(f, f) \in CU_+ \text{ for all } x \in \mathbb{R}_+.$$

Lemma 3.3. *(see Ref. [7]) Let B_+ and B_- be the matrices defined in Eqs. (23). Then*
i) condition (26) is fulfilled, if

$$C^T B_+ C < B_- \text{ on } R\ X_+;$$

ii) condition (25) is fulfilled, if

$$C^T B_+ C \leq B_- \text{ on } R\ U_+.$$

Proof. ii) Let $u \in U_+$ and $C^T B_+ C \leq B_-$ on $R\ U_+$. Then

$$\langle u, u \rangle_B > 0.$$

Furthermore, if $u \neq 0$ and $Cu = 0$, then

$$\langle u, u \rangle_B = \langle Cu_-, Cu_- \rangle_{B_+} - \langle u_-, u_- \rangle_{B_-} = \left\langle (C^T B_+ C - B_-) u_-, u_- \right\rangle \leq 0.$$

Hence, if $Cu = 0$, then $u = 0$. That is, $\dim CU_+ = \dim U_+ = m^+$, and part ii) of the lemma is proved.

Part i) of the lemma is proved in a similar way (see also Ref. [7]). □

Corollary 1. *(see Ref. [7]) If $C = 0$, then the conditions (25) and*

where

$$\begin{cases} {}_i(x) = {}_i(0)\, e^{\quad x}, \quad i = 1, \ldots \end{cases}$$

where

$$\begin{cases} {}_i(f(x)) = {}_i(f(0))e^{\ x}; \ i = 1;:::;k^+, \\ {}_j(f(x)) = {}_j(f(0))e^{\ x} + \int_0^x e^{(\ \ x)} e_j(f(\ ))\,d\ , \ j = 1;:::;l, \\ {}_r(f(x)) = {}_r(f(0))e^{\ _r x} + \int_0^x e^{(\ \ x)\ _r} e_r(f(\ ))\,d\ , \ r = 1;:::;m^+; \\ {}_r(f(x)) = \int_x^1 e^{(\ \ x)\ _r} e_r(f(\ ))\,d\ ; \ r = m^+ + 1;:::;q; \end{cases}$$

with ${}_1(f(0));:::;{}_{m^+}(f(0));\ {}_1(f(0));:::;{}_{k^+}(f(0))$ and ${}_1(f(0));:::;{}_l(f(0))$ given by the system

$$\sum_{r=1}^{m^+} {}_r(f(0))Cu_r + \sum_{i=1}^{k^+} {}_i(f(0))Cy_i + \sum_{j=1}^{l} {}_j(f(0))Cz_j$$

$$= h_0 + \sum_{r=m^++1}^{q} \int_0^1 e^{\ _r} e_r(f(\ ))\,d\ Cu_r,$$

and

$$C = R_+ \ \ CR\ , \ e_j(f) = hS(f;f);w_j i, \text{ and } e_r(f) = hS(f;f);u_r i:$$

Lemma 5.2. *Let $f;h \in X$ and assume that the condition (26) is fulfilled. Then there is a positive constant K (independent of f and h), such that*

$$j\ (0)j \quad K\,jh_0j, \tag{44}$$

$$j\ (f)\ \ (h)j \quad K(jfj + jhj)\,jf\ \ hj\ . \tag{45}$$

Proof. Let C^{-1} denote the inverse map of the linear map $C = R_+ \ CR$ on $X_+ = \text{span}(u_1;:::;u_{m^+};y_1;:::;y_{k^+};z_1;:::;z_l):$ The map C^{-1} is also linear and therefore bounded. We denote

$$P = (u_1:::u_q y_1:::y_k z_1:::z_l w_1:::w_l)$$

(cf. Eqs. (17)-(19)). By Eq. (19)

$P^{-1} = D^{-1}\widetilde{P}^t B$, where

$\widetilde{P} = (u_1:::u_q y_1:::y_k z_1:::z_l w_1:::w_l)$ and $D = \text{diag}(\ _1:::\ _q\ \ _1:::\ _k 1:::1)$.

Then

$j\ (0)j =$

Then

d=1[(t)]TJ/F1 1 9.926 Tf 3.507 6
1820Td[(:::)]TJ/F89.96 Tf 13.5 Td[(1)]TJ 9.926 Tf 0.516 Td]TJ/F12.5 12 4.396 Td[(1)]TJ/F9 5.97

and

$$
\begin{aligned}
& C^{\ 1}C \sum_{r=m^{+}+1} \int_0^{\infty} e^{\ \ r} \left(\tilde{e}_r(f(\ )) \quad \tilde{e}_r(h(\ )) \right) d \ u_r \\
& C^{\ 1}CP \int_0^{\infty} e \sum_{r=m^{+}+1} \left(\tilde{e}_r(f(\ )) \quad \tilde{e}_r(h(\ )) \right) P^{\ 1}u_r \, d \\
& C^{\ 1}CP \int_0^{\infty} e^{\ 3} \ d \quad P^{\ 1}B^{\ 1}(S(f;f) \quad S(h;h))_{\ 2} .
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
& j \ (f) \quad (h)j \\
= \ & PP^{\ 1}(\ (f) \quad (h)) \quad jPj \ P^{\ 1}(\ (f) \quad (h)) \\
& jPj \sup_{x \ 0} \left(\int_x^{\infty} e^{(2x \ \ )} \ P^{\ 1} \sum_{r=m^{+}+1} (\tilde{e}_r(f(\ )) \quad \tilde{e}_r(h(\ )))u_r \ d \right. \\
& \quad + \int_0^{x} e \quad P^{\ 1} \left(\sum_{r=1}^{m^{+}} (\tilde{e}_r(f(\ )) \quad \tilde{e}_r(h(\ )))u_r \right. \\
& \quad \left. \left. + \sum_{j=1} (\tilde{e}_j(f(\ )) \quad \tilde{e}_j(h(\ )))z_j \right) d \right) \\
& \quad + P^{\ 1} \left(\sum_{r=1}^{m^{+}} (\ _r(f(0)) \quad _r(h(0)))u_r + \sum_{j=1} (\ _j(f(0)) \quad _j(h(0)))z_j \right. \\
& \quad \left. + \sum_{i=1}^{k^{+}} (\ _i(f(0)) \quad _i(h(0)))y_i) \right) \\
& jPj (\sup_{x \ 0} \int_x^{\infty} e^{(2x \ \ 3 \ )} \ d \ + \int_0^{\infty} e \quad d \ + P^{\ 1} \quad C^{\ 1}CP \int_0^{\infty} e^{\ 3} \quad d \ ) \\
& \quad P^{\ 1}B^{\ 1}(S(f;f) \quad S(h;h))_{\ 2} \\
& K_1 jS(f;f) \quad S(h;h)j_2 \ , \text{ with } K_1 = \frac{1}{3} jPj \ D^{\ 1} P^{et} \quad 4 + \ P^{\ 1} \quad C^{\ 1}CP \quad .
\end{aligned}
$$

then the system (36) *with the boundary conditions* (21) *and* (24) *has a unique solution* $f = f(x)$ *in* S_R *for a suitable chosen* R*.*

Proof. By estimates (44) and (45), there is a positive number K such that

$$|\ (f)| = |\ (f) \quad (0) + \quad (0)| \quad K(|h_0| + |f|^2) \tag{46}$$

if $f \in X$.

Let $R = 2K$ and let $_0$ be a positive number, such that $_0 < \frac{1}{R^2}$. By estimates (45) and (46)

$$|\ (f)| \quad (\frac{1}{2} + 2K^2 |h_0|) R|h_0| \quad R|h_0|$$

and

$$|\ (f) \quad (h)| \quad 2KR|h_0||f \quad h| \quad R^2 \ _0 |f \quad h| \ , R^2 \ _0 < 1,$$

if $f, h \in S_R$ and $|h_0| \quad _0$.

The theorem follows by the contraction mapping theorem (see Ref. [33, p.2]). □

Theorem 5.4. *Suppose that* $\langle S(f, f), w_j \rangle = 0$ *for* $j = 1, ..., l$*. Then the solution of Theorem* 5.3 *is a solution of the problem* (20)*,* (21) *and* (24) *if and only if* $P_0^+ f(0) = 0$*.*

Proof. The relations

$$\begin{aligned} _i(f(x)) &= \ _i(f(0))e^{\quad x}, \ i = 1, ..., k^+, \\ _j(f(x)) &= \ _j(f(0))e^{\quad x}, \ j = 1, ..., l, \end{aligned}$$

are fulfilled if $f(x)$ is a solution of Theorem 5.3 and $\langle S(f, f), w_j \rangle = 0$. Hence, $P_0^+ f(0) = 0$ if and only if $P_0^+ f(x) \quad 0$. □

We denote by Γ the linear solution operator

$$\Gamma(h_0) = f(0),$$

where $f(x)$ is given by

$$\begin{cases} B\dfrac{df}{dx} + Lf + \quad P_0^+ f = 0 \\ Cf(0) = h_0 \\ f \to 0, \text{ as } x \to \infty \end{cases}.$$

Similarly, we denote by I the nonlinear solution operator

$$I(h_0) = f(0),$$

where $f(x)$ is given by

$$\begin{cases} B\dfrac{df}{dx} + Lf = S(f, f) \quad P_0^+ f \\ Cf(0) = h_0 \\ f \to 0, \text{ as } x \to \infty \end{cases}.$$

We assume that $\langle S(f, f), w_j \rangle = 0$ for $j = 1, ..., l$. By Theorem 5.4, the solution of Theorem 5.3 is a solution of the problem (20), (21) and (24) if and only if $P_0^+ I(h_0) \quad 0$.

Let

$$r_i = \frac{r_i'}{\sqrt{\langle r_i', r_i' \rangle_{B_+}}};$$

$$\text{with } r_i' = Cy_i - \sum_{r=1}^{m^+} \frac{\langle Cy_i, Cu_r \rangle_{B_+}}{\langle Cu_r, Cu_r \rangle_{B_+}} Cu_r - \sum_{j=1}^{i-1} \langle Cy_i, r_j \rangle_{B_+} r_j \neq 0,\ i = 1, \dots, k^+;$$

and

$$r_{i+k^+} = \frac{r_{i+k^+}'}{\sqrt{\left\langle r_{i+k^+}', r_{i+k^+}' \right\rangle_{B_+}}};$$

$$\text{with } r_{i+k^+}' = Cz_i - \sum_{r=1}^{m^+} \frac{\langle Cz_i, Cu_r \rangle_{B_+}}{\langle Cu_r, Cu_r \rangle_{B_+}} Cu_r - \sum_{j=1}^{i+k^+-1} \langle Cz_i, r_j \rangle_{B_+} r_j \neq 0,\ i = 1, \dots, l.$$

Then

$$P_0^+ I(0, h_0),\quad h_0 \in R^{\perp_{B_+}};$$

$$\text{with } R^{\perp_{B_+}} = \left\{ u \in \mathbb{R}^{n^+} \mid \langle u, r_i \rangle_{B_+} = 0 \text{ for } i = 1, \dots, k^+ + l \right\}$$

and

$$I(h_0) \equiv \Phi(a_1, \dots, a_{k^++l}, h_1);$$

$$\text{with } h_0 = \sum_{i=1}^{k^++l} a_i r_i + h_1,\ h_1 \in R^{\perp_{B_+}} \text{ and } a_i = \langle h_0, r_i \rangle_{B_+}.$$

Lemma 5.5. *Suppose that $P_0^+ I(h_0) \equiv 0$. Then h_0 is a function of h_1 if $\langle h_0, h_0 \rangle_{B_+}$ is sufficiently small.*

Proof. It is obvious that $I(0) = 0$ and that we for the Frechet derivative of $I(\varepsilon h_0)$ have

$$\frac{d}{d\varepsilon} I(\varepsilon h_0) \Big|_{\varepsilon=0} = \mathrm{I}(h_0).$$

Then if $h_0 = r_i$

$$\frac{\partial}{\partial a_i} \left\langle \Phi(a_1, \dots, a_{k^++l}, h_1), u \right\rangle_B \Big|_{a_i=0} = \frac{d}{d\varepsilon} \langle I(\varepsilon h_0), u \rangle_B \Big|_{\varepsilon=0} = \langle \mathrm{I}(h_0), u \rangle_B \neq 0,$$

where $u = y_i$ if $i = 1, \dots, k^+$ and $u = w_{i-k^+}$ if $i = k^+ + 1, \dots, k^+ + l$. By the implicit function theorem, $\left\langle \Phi(a_1, \dots, a_{k^++l}, h_1), y_1 \right\rangle_B = 0$ defines $a_1 = a_1(a_2, \dots, a_{k^++l}, h_1)$. Induction gives that

$$a_1 = a_1(h_1), \dots, a_{k^++l} = a_{k^++l}(h_1).$$

□

6. **Direct approach without damping term.** In this section we deal directly Tf 3.87+6.

with the boundary conditions (21) have (under the assumption that all necessary integrals exist) the general solution

$$f(x) = \sum_{j=1}^{l} \ _j(x) z_j + \sum_{r=1}^{q} \ _r(x) u_r, \tag{48}$$

where

$$\ _j(x) = \int_x^{\infty} e_j(\ )\, d\ , \quad e_j(x) = \langle g(x); w_j \rangle, \ j = 1, \ldots, l, \tag{49}$$

and $_1(x), \ldots, \ _q(x)$ are given by Eq. (40). From the boundary conditions (24), we obtain the system

$$C \sum_{r=1}^{m^+} \ _r(0) u_r = h_0 + C \int_0^{\infty} \sum_{r=m^++1}^{q} e^{\ r} e_r(\ ) u_r + \sum_{j=1}^{l} e_j(\ ) z_j \, d\ ; \tag{50}$$

with $C = R_+ \ CR$.

The system (50) has (under the assumption that all necessary integrals exist) a solution if we assume that

$$h_0, Ce^{xB^{-1}L} B^{-1} g(x) \in CU_+ \text{ for all } x \in \mathbb{R}_+,$$

$$\text{with } U_+ = \text{span}(u : Lu = \ Bu, \ > 0) = \text{span}(u_1, \ldots, u_{m^+}) \tag{51}$$

and a unique solution if and only if, additionally, condition (25) is fulfilled.

Theorem 6.1. *Assume that the conditions* (25), (38) *and* (51) *arF44 F44 (s)]TJ/F8 9.9626 Tf 5.858 -1.32 Td [(j)]TJ/F8 9.9626 Tf 0.858 1.32 Td[(...

Lemma 7.1. *Let* $g(x) \in N(L)^{\perp}$ *and assume that*

$$N(L) \cap N(B) = \{0\}.$$

Then the linear operators $\tilde{L}_{\mathrm{Im}}$ *and* $\tilde{B}_{\mathrm{Im}}$ *on* $\mathrm{Im}(B)$ *have the following properties:* $\tilde{L}_{\mathrm{Im}}$ *and* $\tilde{B}_{\mathrm{Im}}$ *are real symmetric operators,* $\tilde{L}_{\mathrm{Im}}$ *is semi-positive,* $\tilde{B}_{\mathrm{Im}}$ *is non-singular,* $\dim(N(\tilde{L}_{\mathrm{Im}})) = p$*, and the numbers* k^{+}, k^{-} *and* l *are the same for the system*

$$\tilde{B}_{\mathrm{Im}} \frac{dP_1 f}{}$$

L_{Im})](()]TJ/F11 9.9626 Tf 3.875 0 Td [(B)]TJ/F8 9.9626 Tf 5.694 0 Td [())]TJ/F14 9.9626 Tf 6.733 0 Td [(L)]TJ/F8 9.9626 Tf -291.339 1044

Denote

$$\widetilde{\ }(f) = \quad (P_1 f);$$

where is the operator (43) when L and $S(f; f)$ are replaced with $\widetilde{L}$ and $\widetilde{S}(f; f) = P_1(I \quad LP_0 (P_0LP_0)^{\ 1} P_0)S(f; f)$ in Eq. (36). We introduce

$$\widehat{\ }(f) = \quad I \quad (P_0LP_0)^{\ 1} P_0L \quad \widetilde{\ }(f) + (P_0LP_0)^{\ 1} P_0S(f; f)$$

and denote

$$_{\min} = \min j \ _i j \text{ and } \ _{\max} = \max j \ _i j ,$$

where $_1; :::; \ _{n \ p}$ are the non-zero eigenvalues of L. Then

$$\widehat{\ }(0) \quad = \quad I \quad (P_0LP_0)^{\ 1} P_0L \quad \widetilde{\ }(0) \qquad \widehat{K}_0 jh_0j , \text{ with } \widehat{K}_0 = (1 + \ \tfrac{1}{\min} \ _{\max})\widetilde{K} ,$$

and

$$\widehat{\ }(f) \quad \widehat{\ }(g)$$

$$= \quad I \quad (P_0LP_0)^{\ 1} P_0L \quad \widetilde{\ }(f) \quad \widetilde{\ }(g) \quad + (P_0LP_0)^{\ 1} P_0 (S(f; f) \quad S(g; g))$$

$$(1 + \ \tfrac{1}{\min} \ _{\max})\widetilde{K}(jf j \ + jhj \ )jf \quad hj \quad + \ \tfrac{1}{\min}K_2(jf j \ + jhj \ )jf \quad hj$$

$$= \widehat{K}_1(jf j \ + jhj \ )jf \quad hj \ , \text{ with } \widehat{K}_1 = \widetilde{K} + \ \tfrac{1}{\min}(\widetilde{K} \ _{\max} + K_2).$$

We can now extend our main results in Section 3 to yield also for singular operators B.

8. Axially symmetric DVMs. In this section we consider only such symmetric sets of velocities V, such that

$$\text{if } \ _i = (\ ^1_i; :::; \ ^d_i) \ 2 \ \mathcal{V}; \text{ then } (\quad ^1_i; :::; \quad ^d_i) \ 2 \ \mathcal{V} \tag{54}$$

for any combinations of signs (see also Ref. [7]). We can, without loss of generality, assume that

$$(\ ^1_{i+N}; \ ^2_{i+N}; :::; \ ^d_{i+N}) = (\quad ^1_i; \ ^2_i; :::; \ ^d_i) \text{ and } \ ^1_i > 0;$$

for $i = 1; :::; N$, with $n = 2_{i \ j}$

normal model with velocities (1; 1; 1) and (3; 1; 1). The 24-velocity models, with velocities (1; 1; 1), (1; 3; 1) and (3; 1; 1), and (1; 1; 1), (1; 1; 3) and (3; 1; 1), respectively, are DVMs with fewer velocities (earlier in the "evolution"), that can be constructed from the same asymmetric model.

8.1. **Explicit calculation of the characteristic numbers.** We now assume that (i) we have a symmetric set (54) of velocities; (ii) our DVM is normal; (iii) we have made the expansion (13) around a non-drifting Maxwellian M, i.e. with $\mathbf{b} = \mathbf{0}$ in Eq. (12); and (iv)

$$B = \mathrm{diag}({}^1_1 ; :::; {}^1_N ; \quad {}^1_1 ; :::; \quad {}^1_N), \text{ with } {}^1_1 ; :::; {}^1_N > 0.$$

In this section we study, instead of Eq. (20), the equation

$$(B + uI)\frac{df}{dx} + Lf = S(f; f), \tag{55}$$

(cf. Eq. (3)). Note, however, that Eqs. (20) and (55) are never equivalent for non-zero u, as Eqs. (2) and (3) are in the continuous case, for DVMs with a nite number of velocities.

The linearized collision operator L has the null-space

$$N(L) = \mathrm{span}({}_1 ; :::; {}_{d+2}),$$

where

$$\begin{cases} {}_1 = M^{1=2} (1; :::; 1) \\ {}_2 = M^{1=2} ({}^1_1 ; :::; {}^1_N ; \quad {}^1_1 ; :::; \quad {}^1_N) \\ {}_3 = M^{1=2} (j \ {}_1 j^2 ; :::; j \ {}_N j^2 ; j \ {}_1 j^2 ; :::; j \ {}_N j^2) \\ {}_{i+2} = M^{1=2} ({}^i_1 ; :::; {}^i_N ; \quad {}^i_1 ; :::; \quad {}^i_N), \ i = 2; :::; d; \end{cases} . \tag{56}$$

Then the degenerate values of u, i.e. the values of u for which l 1, are

$$u_0 = 0 \text{ and } u = \sqrt{\frac{ {}_1 \ {}^2_4 + \ {}^2_2 \ {}_5 \quad 2 \ {}_2 \ {}_3 \ {}_4 }{2(\ {}_1 }}$$

Remark 8. For the continuous Boltzmann equation (with $d = 3$) the numbers $_1; \dots; \ _5$ are given by

$$_1 = \ , \quad _2 = \ T, \quad _3 = 3 \ T, \quad _4 = 5 \ T^2 \text{ and } \quad _5 = 15 \ T^2,$$

(where   and T denote the density and the temperature respectively), if we have made the expansion (13) around a non-drifting Maxwellian

$$M = \frac{}{(2 \ T)^{3/2}} e^{\ j \ j^2/2T}.$$

Therefore, for the Boltzmann equation (with $d = 3$) the degenerate values (57) are (cf. Ref. [21])

$$u_0 = 0 \text{ and } u \ = \ \sqrt{\frac{5T}{3}}.$$

Below we return to study Eq. (20).

8.2. **Plane 12-velocity model.** For $d = 2$ the equations (5) admit a class of solutions satisfying

$$F_i = F_{i'} \text{ if } \ _i^1 = \ _{i'}^1 \text{ and } j \ _i j^2 = j \ _{i'} j^2. \tag{58}$$

This reduces the number n of equations (5) to the number $2N < n$ of di erent combinations $(\ _i^1; j \ _i j^2)$ in the velocity set. However, the structure of the collision terms (7) (in slightly di erent notations) remains unchanged. We can, without loss of generality, assume that

$$(\ _{i+N}^1; j \ _{i+N} j^2) = (\ \ _i^1; j \ _i j^2) \text{ and } \ _i^1 > 0$$

for $i = 1; \dots; N$. Then, the Maxwellians are of the form

$$M_i = Ae^{b \ _i^1 + cj \ _i j^2} = M_{i+N} e^{2b \ _i^1}, \ i = 1; \dots; N;$$

for some constant $A; b; c \in \mathbb{R}$, with $A > 0$.

For the 12-velocity model in Example 3 (see Ref. [7]), the system (5) reduces by reduction (58) to a system of the form

$$\begin{cases} \dfrac{dF_1}{dx} = \ _1 q_1 + \ _2 q_2 + \ _3 q_3 \\ \dfrac{dF_2}{dx} = \ _1 q_1 \quad \ _2 q_2 + \ _4 q_4 \\ 3\dfrac{dF_3}{dx} = \ (\ _1 q_1 + \ _4 q_4) \\ \dfrac{dF_4}{dx} = \ (\ _1 q_1 + \ _2 q_2 + \ _3 q_3) \\ \dfrac{dF_5}{} \end{cases}$$

where $s = e^{4c}$ and $K = Ae^{2c}$, c and A are constant, with $A > 0$. The null-space of L is given by

$$N(L) = \text{span}\,({}_1; {}_2; {}_3)\,,$$

where

$$\begin{cases} {}_1 = K^{1=2}\,(1; s; s; 1; s; s) \\ {}_2 = K^{1=2}\,(1; s; 3s; \;\; 1; \;\; s; \;\; 3s) \\ {}_3 = 2K^{1=2}\,(1; 5s; 5s; 1; 5s; 5s) \end{cases}$$

and

$$B = \text{diag}(1; 1; 3; \;\; 1; \;\; 1; \;\; 3).$$

A typical choice of ${}_1; {}_2; {}_3; {}_4$ (cf. Refs. [15] and [23]) is

$$\begin{cases} {}_1 = {}_3 = 2S \\ {}_2 = S(\sqrt{2} + \sqrt{5}) \\ {}_4 = S\sqrt{10}; \end{cases}.$$

Therefore, we assume below that ${}_1 = {}_3; {}_2; {}_4 > 0$. Then

$$L = \begin{pmatrix} (2{}_1 + {}_2)s^2 & ({}_1 \;\; {}_2)s & {}_1 s & \\ ({}_1 \;\; {}_2)s & {}_1 + {}_2 + {}_4 s^2 & {}_1 + {}_4 s^2 & \\ {}_1 s & {}_1 + {}_4 s^2 & {}_1 + {}_4 s^2 & \\ (2{}_1 + {}_2)s^2 & ({}_2 \;\; {}_1)s & {}_1 s & \\ ({}_2 \;\; {}_1)s & {}_2 + {}_4 s^2 & {}_4 s^2 & \\ {}_1 s & {}_4 s^2 & {}_4 s^2 & \\ (2{}_1 + {}_2)s^2 & ({}_2 \;\; {}_1)s & {}_1 s & \\ ({}_2 \;\; {}_1)s & {}_2 + {}_4 s^2 & {}_4 s^2 & \\ {}_1 s & {}_4 s^2 & {}_4 s^2 & \\ (2{}_1 + {}_2)s^2 & ({}_1 \;\; {}_2)s & {}_1 s & \\ ({}_1 \;\; {}_2)s & {}_1 + {}_2 + {}_4 s^2 & ({}_1 + {}_4 s^2) & \\ {}_1 s & ({}_1 + {}_4 s^2) & {}_1 + {}_4 s^2 & \end{pmatrix}$$

and

$$S(f; f) = \begin{pmatrix} s\,{}_1(q_1 + q_3) + s\,{}_2 q_2 \;\; {}_2 {}_4 s^2 \; {}_1 + q_2 \\ \vdots \end{pmatrix}$$

then

$$y_1; y_2; z \in N(L); \; Lw = Bz,$$
$$\langle y_1; y_2 \rangle_B = \langle z; z \rangle_B = \langle w; w \rangle_B = \langle z; y_i \rangle_B = \langle w; y_i \rangle_B = 0 \text{ for } i = 1; 2,$$
$$\langle y_1; y_1 \rangle_B = \quad \langle y_2; y_2 \rangle_B = 16 \text{ and } \langle z; w \rangle_B = \frac{18}{\ _1} + \frac{16}{\ _2}.$$

Furthermore, if

$$u_1 = \sqrt{2}\tilde{e}_1 + [2\sqrt{2} \ _1 \quad \sqrt{(8 \ _1 + 9 \ _2)(\ _1 + 2 \ _4 s^2)}]\tilde{e}_2$$
$$u_2 = \sqrt{2}\tilde{e}_1 + [2\sqrt{2} \ _1 + \sqrt{(8 \ _1 + 9 \ _2)(\ _1 + 2 \ _4 s^2)}]\tilde{e}_2 \text{ , with}$$

$$\tilde{e}_1 = (3s(3 \ _2 \quad 4 \ _1); \quad 3(4 \ _1 + 3 \ _2); 4 \ _1; 3s(3 \ _2 \quad 4 \ _1); \quad 9 \ _2; 0) \text{ and}$$
$$\tilde{e}_2 = (0; 3; \quad 1; 0; \quad 3; 1),$$

then

$$Lu_i = \ _i Bu_i \text{ for } i = 1; 2, \text{ with } \ _1 = \quad _2 = \frac{\sqrt{2(8 \ _1 + 9 \ _2)(\ _1 + 2 \ _4 s^2)}}{3},$$
$$\langle u_1; u_2 \rangle_B = \langle u_i; z \rangle_B = \langle u_i; w \rangle_B = \langle u_i; y_j \rangle_B = 0 \text{ for } i; j = 1; 2, \text{ and}$$
$$\langle u_1; u_1 \rangle_B = \quad \langle u_2; u_2 \rangle_B = 12(8 \ _1 + 9 \ _2) \sqrt{2(8 \ _1 + 9 \ _2)(\ _1 + 2 \ _4 s^2)}.$$

We have that

$$(R_+ \quad R \ )u_1 = \quad 2\sqrt{(8 \ _1 + 9 \ _2)(\ _1 + 2 \ _4 s^2)} R_+ \tilde{e}_2 = \quad (R_+ \quad R \ )u_2 \text{ and}$$
$$(R_+ \quad R \ )z = 0$$

and so Theorem 3.2 is applicable for $C = \mathrm{diag}(1; 1; 1)$ (C = the identity operator) and $h_0 \in \mathrm{span}((0; 3; \quad 1))$ sufficiently small (cf. Refs. [2] and [26]).

8.3. **More general axially symmetric DVMs.** Now, additionally to assumptions (i)-(iv) above, we assume that (v) the coefficients $\ _{ij}^{kl}$ in Eq. (7) satisfy the additional symmetric conditions

$$\ _{ij}^{kl} = \quad ^{(k)(l)}_{(i)(j)},$$

where $\ (i) = \begin{cases} i + N, \text{ if } 1 \quad i \quad N, \\ i \quad N, \text{ if } N + 1 \quad i \quad 2N, \end{cases}$.

Then $L = \begin{pmatrix} L_1 & L_2 \\ L_2 & L_1 \end{pmatrix}$, where L_1 and L_2 are two $N \quad N$ matrices (cf. Refs. [2], [3], [4] and [7]). We choose

$$\begin{cases} \varphi_1 = \ _2 + \ _3 \\ \varphi_2 = \ _2 \quad _3 \\ \varphi_3 = \ _4 \ _1 \quad _2 \ _3 \end{cases}.$$

where $\ _2 = \langle \ _2; \ _2 \rangle$, $\ _4 = \langle \ _2; \ _3 \rangle_B$ and $\ _1; \ _2; \ _3$ are given in Eq. (56). Then

$$K = 2 \ _2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

where $K = (\langle \varphi_i; \varphi_j \rangle_B)$. Hence, $k^+ = k \quad = 1$ and $l = d$, since $\ _4; \ldots; \ _{d+2}$ are all orthogonal, with respect to the scalar product $\langle \cdot; \cdot \rangle_B$, to $\ _1; \ _2$ and $\ _3$. Since

$B^{-1}L\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \lambda\begin{pmatrix} u^+ \\ u^- \end{pmatrix}$, with $u^+, u^- \in \mathbb{R}^N$, implies that $B^{-1}L\begin{pmatrix} u^- \\ u^+ \end{pmatrix} = -\lambda\begin{pmatrix} u^- \\ u^+ \end{pmatrix}$, we obtain (cf. Refs. [2], [3], [4] and [7]) that the non-negative eigenvalues of $B^{-1}L$ are $\lambda_1, ..., \lambda_{N-l-1}$, where $\lambda_i > 0$, with corresponding eigenvectors $u_1, ..., u_{N-l-1}$, where $R_+u_i = R_-u_i^+$ and $R_-u_i = R_+u_i^+$.

Therefore, the Jordan normal form of $B^{-1}L$ for $d = 3$ is (the number of blocks $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is equal to the dimension d, that is, in this case 3)

$$\begin{pmatrix} \lambda_1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & \lambda_{m^+} & & & & & & & \\ & & & -\lambda_1 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & -\lambda_{m^+} & & & & \\ & & & & & & 0 & & & \\ & & & & & & & 0 & & \\ & & & & & & & & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & & \\ & & & & & & & & & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \\ & & & & & & & & & & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

If $C = \mathrm{diag}(1, ..., 1)$, i.e. if C is the identity operator, and $h_0 \in (R_+ - R_-)\mathrm{span}\left(u_1^+, ..., u_{N-l-1}^+\right)$, then the conditions (25) and (28) are fulfilled (see Remark 5).

Under the assumptions (i)-(v) given above, the following theorem (see Ref. [26] for the case of the continuous Boltzmann Equation) follows by Theorem 3.2.

Theorem 8.1. *Let $h_0 \in (R_+ - R_-)U_+$, where $U_+ = \mathrm{span}\left(u_1^+, ..., u_{N-l-1}^+\right)$. Then there is a positive number δ_0, such that if*

$$|h_0| \leq \delta_0,$$

then the system (20) *with the boundary conditions*

$$f(x) \to 0, \text{ as } x \to \infty, \text{ and } (R_+ - R_-)f(0) = h_0,$$

has a locally unique solution $f = f(x)$.

Remark 9. The same problem, for $d = 2$, is also studied by Babovsky in Ref. [2], but then under the quite restrictive condition $\langle S(f, f), w_i \rangle = 0$ for $i = 1, 2$ (in our notations).

Acknowledgments. I thank my former advisor professor Alexander Bobylev for introducing me to this problem and giving me part of his great ideas and knowledge.

REFERENCES

[1] K. Aoki, C. Bardos and S. Takata *Knudsen layer for gas mixtures*, J. Stat. Phys., **112** (2003), 629–655.

[2] H. Babovsky, *Kinetic boundary layers: on the adequate discretization of the Boltzmann collision operator*, J. Comput. Appl. Math., **110** (1999), 225–239.

[3] H. Babovsky and P. Kowalczyk, *Di usion limits for discrete velocity models in a thin gap*,

[33] D. R. Smart, \Fixed Point Theorems" Cambridge University Press, 1974.

[34] Y. Sone, \Kinetic Theory and Fluid Dynamics" Birkhuser, 2002.

[35] Y. Sone, \Molecular Gas Dynamics" Birkhuser, 2007.

[36] Q. Tian, *Existence of nonlinear boundary layer solution to the Boltzmann equation with physical boundary conditions*, J. Math. Anal. Appl., **356** (2009), 42{59.

[37] S. Ukai, *On the half-space problem for the discrete velocity model of the Boltzmann equation*, in \Advances in Nonlinear Partial Di erential Equations and Stochastics" (eds. S. Kawashima and T. Yanagisawa), World Scienti c, (1998), 160{174.

[38] S. Ukai, T. Yang and S-H. Yu, *Nonlinear boundary layers of the Boltzmann equation. I. Existence*, Comm. Math. Phys., **236** (2003), 373{393.

[39] S. Ukai, T. Yang and S-H. Yu, *Nonlinear stability of boundary layers of the Boltzmann equation. I. The case* $\mathcal{M}^{1} < -1$, Comm. Math. Phys., **244** (2004), 99{109.

[40] W. Wang, T. Yang and X. Yang, *Nonlinear stability of boundary layers of the Boltzmann equation for cuto hard potentials*, J. Math. Phys., **47** (2006), 083301:1{15.

[41] W. Wang, T. Yang and X. Yang, *Existence of boundary layers to the Boltzmann equation with cuto soft potentials*, J. Math. Phys., **48** (2007), 073304:1{21.

[42] X. Yang, *Nonlinear stability of boundary layers for the Boltzmann equation with cuto soft potentials*, J. Math. Anal. Appl., **345** (2008), 941{953.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: niclas.bernhoff@kau.se