Discrete Quantum Boltzmann Equation

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for bosons, and = 1, corresponding to the discrete Nordheim-Boltzmann equation for fermions. Unlike for the intermediate cases, in the limiting cases for bosons and fermions, the three dimensional case d = 3 is highly relevant

The inequality (6) is obtained by using the relation

$$(z - y)\log\frac{y}{z} \leq 0,$$

with equality if and only if $y = z$, which is valid for all $y; z \in \mathrm{R}_+$. Hence, we have equality in the inequality (6) if and only if

$$\frac{F_i}{(F_i)}\frac{F_j}{(F_j)} = \frac{F_k}{(F_k)}\frac{F_l}{(F_l)}, \qquad (7)$$

for all indices such that

$\mathcal{H}$-THEOREM AND TREND TO EQUILIBRIUM

We define an $\mathcal{H}$-function

$$\mathcal{H}[F] = \mathcal{H}[F](x) = \sum_{i=1} p_i^1 \ \ (F_i(x));$$

where, cf. [8],

$$(y) = \begin{cases} y \log y + (1 - \ \ y) \log (1 - \ \ y) - (1 + (1 - \ \ ) y) \log (1 + (1 - \ \ ) y) \ \text{ if } 0 < y < \frac{1}{\ } \\ 0 \text{ if } y = 0 \text{ or (if } \ \ \neq 0) \ y = \frac{1}{\ } \end{cases} : \qquad (12)$$

For the planar stationary system

$$B\frac{dF}{dx} = Q \ \ (F) \text{, with } B = \text{diag}(p_1^1; ::::; p_N^1), \ x \in \mathbb{R}_+, \qquad (13)$$

we obtain an $\mathcal{H}$-theorem

$$\frac{d}{dx}\mathcal{H}[F] = \sum_{i=1} p_i^1 \frac{dF_i}{dx} \log \frac{F_i}{\ \ (F_i)} = \left\langle \log \frac{F}{\ \ (F)}; Q \ \ (F) \right\rangle \leq 0,$$

with equality if, and only if, F is an equilibrium distribution. We introduce the moments

$$\begin{cases} j_1 = \langle B\mathbf{1}; F \rangle \\ j_{i+1} = \left\langle Bp^i; F \right\rangle, \ i = 1; ::::; d \\ j_{d+2} = \left\langle B\,|\mathbf{p}|^2 ; F \right\rangle \end{cases}, \ \mathbf{1} = (1; ::::; 1) \in \mathbb{R}^N: \qquad (14)$$

By applying relation (10) for the system (13), one obtain that $j_1; ::::; j_{d+2}$ are independent of x in the planar stationary case. For some fixed $j_1; ::::; j_{d+2}$, we denote by P the manifold of all equilibrium distributions $F = P$ (given by equation (11)) with the moments (14). Then one can show the following theorem by arguments similar to the ones used for the discrete Boltzmann equation in [17] (or, also [18]).

Theorem 1 *If $F = F(x)$ is a solution to the system* (13)*, such that $0 \leq F_i \leq \frac{1}{\ }$ (F_i are non-negative bounded functions for* $\ \ = 0$*), then*

$$\lim_{x\to\infty} \text{dist}(F(x); \text{P}) = 0;$$

where P *is the manifold of equilibrium distributions associated with the invariants* (14) *of F. If there are only finitely many equilibrium distributions in* P*, then there is an equilibrium distribution P in* P*, such that $\lim_{x\to\infty} F(x) = P$.*

Remark 2 *A key point in the proof of Theorem* 1 *(as well as - with x replaced with t - for Theorem* 2 *below), cf. [17],[18], is that*

$$\int_0^\infty \frac{d}{dx}\mathcal{H}[F] \, dx = \lim_{x\to\infty}$$

For the spatially homogeneous system

dF

and we will have to exclude the moments j_1 *and* $\tilde{j}_1$ *from the moments* (14) *and* (16)*, respectively, for Theorem* 1 *and Theorem* 2 *to stay valid. A drawback is that, in general, it will not be clear how to construct the sets* $\mathcal{P}$ *to obtain normal DVMs. An example when such generalizations (with* $= 0$*) are of interest is for excitations in a Bose gas* since, by the relations (20) and (21),

$$\frac{(P_i) - \ ' \ (P_i)\, P_i}{P_i \quad (P_i)} R_i = 1. \tag{24}$$

By the relations (4) and (11), we obtain the relation

$$P_i P_j \quad (P_k) \quad (P_l) = P_k P_l \quad (P_i) \quad P_j \tag{25}$$

for $\ ^{kl}_{ij} \neq 0$. and hence, by relations (3), (22), (23), and (25), we obtain the equality

$$\langle g; L f\rangle = \frac{1}{4} \sum_{i;j;k;l=1} \ ^{kl}_{ij} P_i P_j \quad (P_k) \quad (P_l) \left(\frac{f_i}{R_i^{1/2}} + \frac{f_j}{R_j^{1/2}} - \frac{f_k}{R_k^{1/2}} - \frac{f_l}{R_l^{1/2}} \right) \left(\frac{g_i}{R_i^{1/2}} + \frac{g_j}{R_j^{1/2}} - \frac{g_k}{R_k^{1/2}} - \frac{g_l}{R_l^{1/2}} \right). \tag{26}$$

Then it is immediate that the matrix L is symmetric and positive semi-definite, i.e.

$$\langle g; L f\rangle = \langle Lg; f\rangle \text{ and } \langle f; L f\rangle \geq 0,$$

for all $g = g(\mathbf{p})$ and $f = f(\mathbf{p})$.

Furthermore, by the relation (26), $\langle f; L f\rangle = 0$ if and only if

$$\frac{f_i}{R_i^{1/2}} + \frac{f_j}{R_j^{1/2}} = \frac{f_k}{R_k^{1/2}} + \frac{f_l}{R_l^{1/2}} \tag{27}$$

for all indices satisfying $\ ^{kl}_{ij} \neq 0$. We denote $f = R^{1/2}$ in equality (27) and obtain the relation (8). Hence, since L is semi-positive,

$$L f = 0 \text{ if and only if } f = R^{1/2} \ ,$$

where is a collision invariant (8). Hence, for normal models the null-space $N(L)$ is

$$N(L) = \text{span}\ R^{1/2}; R^{1/2} p^1; \ldots; R^{1/2} p^d; R^{1/2} |\mathbf{p}|^2 \ , R = P(1 - \ P)(1 + (1 - \ )P). \tag{28}$$

Remark 4 *More generally, we can consider the collision operator* (19) *and obtain corresponding results for the linearized collision operator L by similar arguments. In particular, the linearized operator L is symmetric and positive semi-definite. However, if at least one* a_{mn} *such that* $m \neq n$ *is nonzero then for normal models the following holds*

$$N(L) = \text{span}\ R^{1/2} p^1; \ldots; R^{1/2} p^d; R^{1/2} |\mathbf{p}|^2 \ , R = P(1 - \ P)(1 + (1 - \ )P).$$

Linearized Spatially Homogeneous Equation

The Cauchy problem for linearized spatially homogeneous equation reads

$$\frac{df}{dt} + L f = 0, \ f(0) = f_0,$$

for some $f_0 \in \mathbb{R}^N$, and has a bounded solution

$$f(t) = e^{-tL} f_0,$$

such that, for any orthogonal basis $\{y_1; \ldots; y_{d+2}\}$ of the null-space $N(L)$ of L (28),

$$f(t) \to \sum_{i=1}^{d+2} \frac{\langle y_i; f_0\rangle}{\langle y_i; y_i\rangle} y_i, \text{ as } t \to \infty.$$

Linearized Half-Space Problems

The linearized steady system in a slab-symmetry reads

$$B\frac{df}{dx} + Lf = 0,\ f(0) = f_0, \text{ with } B = \mathrm{diag}(p_1^1, \ldots, p_N^1),$$

for some $f_0 \in \mathbb{R}^N$, where

$$\mathbf{x} = (x = x^1, x^2, \ldots, x^d) \text{ and } \mathbf{p} = (p^1, \ldots, p^d).$$

Due to the matrix B, here the situation will be much more intricate than for the spatially homogeneous equation. However, since above, some general properties of the "classical" discrete Boltzmann equation (obtained by letting = 1 in the collision operator (2)), have been shown also for the discrete quantum Boltzmann equation, the general results obtained for linearized half-space problems for the discrete Boltzmann equation obtained in [10, 11] (also cf. [24]) hold also for the discrete quantum Boltzmann equation presented here. We note that the numbers of positive, negative, and zero eigenvalues, respectively, of the symmetric $(d + 2) \times (d + 2)$ matrix K with elements

$$k_{ij} = \left\langle y_i, By_j \right\rangle,$$

where $\{y_1, \ldots, y_{d+2}\}$ is any basis of the null-space $N(L)$ of L (28), is of great importance for these results (see for example [10, 11, 24, 12]).

CONCLUDING REMARKS

We have presented a general discrete velocity model (DVM) of Boltzmann equation for anyons (or Haldane statistics), and considered it in the lines of previous studies of the "classical" discrete Boltzmann equation , see e.g. [10, 11, 12]. As two limiting cases the Nordheim-Boltzmann equation for bosons and fermions are also included. The equilibrium distributions are shown to satisfy a transcendental equation (11) (a corresponding equation for the continuous case was first presented in [16]). In certain cases the equation can be analytically solved; we have presented the results in the simplest cases: for bosons, fermions, and so called semions. By a suitable choice of $\mathcal{H}$-function(s), we obtained $\mathcal{H}$-theorem(s), and, thereby could state trend to equilibrium in the spatially homogeneous, as well as, in the planar stationary case. By linearizing around an equilibrium distribution, in a suitable way, we obtained a linearized operator with similar properties to the linearized operator for the classical discrete Boltzmann operator, i.e. a symmetric, positive semi-definite operator, with a null-space of the same dimension as the vector space of the collision invariants. Then the solution of the Cauchy problem for linearized spatially homogeneous equation was immediate, while the results for the linearized steady half-space problems in a slab-symmetry are more intricate. However, while they are not presented here, the results can be found in [10, 11, 24], where they were presented for linearized operators of other discrete Boltzmann equations, but with similar properties. Note that half-space problems for the non-linear Nordheim-Boltzmann equation (i.e. for bosons and fermions) are considered in [12]. We refer the reader to [12] for the obtained results. However, in the general case there are no such results (for the non-linear equation) yet. We also stress that all results presented here are independent of the (finite) number of velocities considered.

All results are in general not depending on which collision invariants that are assumed, so for example other conserved energies than the assumed kinetic one can be considered. However, in implementations, it might be a difficulty to find "good" velocity sets, not to have extra (spurious) collision invariants in plus to the desired (physical) collision invariants (9). On the other hand, for the collision invariants considered, there are well-known procedures to obtain DVMs without spurious collision invariants, see e.g. [13, 14, 15]. The results can also be generalized to more general collision operators (28), cf. Remarks 3 and 4, where in many cases mass will not be conserved, and therefore one might obtain similar difficulties to find "good" velocity sets as in the case of exchanging energy. On the other hand from a theoretical point of view there is nothing preventing such generalizations.

We also want to stress that the results presented here can be generalized to mixtures, and particles with a discrete number of different energy levels, with the approaches presented in [25, 26, 12, 27].

Finally, up to our knowledge and belief, corresponding results - to the ones presented here for DVMs - are also valid in the case of a continuous velocity variable (with a suitable choice of collision kernels). However, even if some of them - in a clear way - can be obtained (correspondingly) as above, some others will be more demanding to show.

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