Outline

- Introduction to the models
- Literature to averaging
- Introduction to homogenization
- Main results
- Some proofs
- Some remarks

Di usion Reaction model

$$\frac{\partial u^{\varepsilon}}{\partial t}(t,x) = \mathrm{div}\left(A\left(\frac{x}{\varepsilon}\right) \nabla u^{\varepsilon}(t,x)\right) + \left(\frac{x}{\varepsilon}, v^{\varepsilon}(t,x)\right) u^{\varepsilon}(t,x) + f(t,x)$$

$$dv^{\varepsilon}(t,x) = \frac{1}{\varepsilon}(v^{\varepsilon}(t,x) \quad u^{\varepsilon}(t,x))dt + \sqrt{\frac{Q}{\varepsilon}}\, dW(t,x);$$

Di usion Convection model

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Multicontinuum model

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Literature on averaging

1. For stochastic di erential equations:
   - R.Z. Khasminskii, *on the principle of averaging the Itô stochastic di erential equation*, Kybernetika, (1968).
   - A.Yu. Veretennikov, *On the averaging principle for systems of stochastic di erential equations*, Mat. USSR Sb., (1991).
   - M. Freidlin, A. Wentzell, *Averaging principle for stochastic perturbations of multifrequency systems*, Stochastics and Dynamics, (2003).
2. For stochastic partial di erential equations:
   - S. Cerrai, *A Khasminskii type averaging principle for stochastic reaction-di usion equations*, Ann. Appl. Probab., (2009)
   - S. Cerrai, M. Freidlin, *Averaging principle for a class of stochastic reaction-di usion equations*, Probab. Theory Related Fields, (2009).

Introduction to the homogenization

Assume to have a sequence of partial di erential operators L_ε (with oscillating coe cients) and a sequence of solutions u_ε which for a given domain D and source f

$$L_\varepsilon u_\varepsilon = f \quad in\ D \tag{2}$$

complemented by appropriate boundary conditions. If we assume that u_ε converges in some sense to some u, we look for a so-called homogenized operator $\overline{L}$ such that

$$\overline{L}u = f \quad in\ D \tag{3}$$

Passing from (2) to (3) is the homogenization process.

Typically

$$L_\varepsilon u^\varepsilon = -\nabla \cdot \left(a(x, \frac{x}{\varepsilon}) \nabla u^\varepsilon\right).$$

Formally, in order to find the form of $\bar{L}$, one writes the expansion

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \ldots \qquad (4)$$

where each $u_i(x, y)$ is periodic in y. Inserting (4) into (2) leads to a cascade of equations for u_i and averaging wrt to y the equation for u_0 gives (3).

Typically

$$\bar{L}u_0(x) = \quad \nabla \quad (\bar{a}(x)\,\nabla u_0(x));$$

where

$$\bar{a}(x) = \int_Y \langle a(x;y)(I + \nabla_y N);(I + \nabla_y N)\rangle dy;$$

such that N is solution of the cell problem:

$$\mathrm{div}(a(I + \nabla N)) = 0 \quad in Y$$

and $y \to N(x;y)$- is Y periodic.

Now, other arguments are needed to prove the convergence of the sequence u_ϵ 0.9091 fT44 0.8451 fT44.122 21.939 0.636 dv122 21.939 0.636 dv122

Homogenization methods

1. The energy method
2. The two-scale convergence

Literature on homogenization

1. Homogenization of the Stokes problem in perforated domains
   - Sanchez-Palencia (1980) (asymptotic expansion method)
   - L. Tartar (the energy method)
   - G. Allaire (1992) (two scale convergence method)
2. Homogenization of PDEs with random coe cients or stochastic forcing in non perforated domains
   - Bourgeat, A. Mikelic and Wright in (1994)
   - P. A. Raza mandimby, M. Sango, and J. L. Woukeng (2012)
3. Homogenization of SPDEs in perforated domains (at its infancy)
   - W. Wang and J. Duan (2007)
   - H. Bessaih, Y. Efendiev and F. Maris (2015, 2016)

Assumptions

The

For any $\varepsilon > 0$, we denote by

$$A^{\varepsilon} : \mathbb{R}^3 \to \mathbb{R}^{3 \quad 3}; \quad A^{\varepsilon}(x) = A \quad \frac{x}{\varepsilon} \quad ; \tag{9}$$

and

$$^{\varepsilon} : L^2(D) \to L^{1}(D); \quad ^{\varepsilon}(\quad)(x) = \quad \frac{x}{\varepsilon}; \quad (x) \quad : \tag{10}$$

We assume that W)(Bm on a ltered poTd [0 g 0 G /F30 10.9091 TQe

Compare with the existing literature

E. Pardoux, A. L. Piatniski (2003),

In Cerrai-Friedlin (2009), they consider

$$du = [A_1 u + B_1(u, v)]dt + G_1(u, v)dW$$
$$dv = \frac{1}{\epsilon}[A_2 v + B_2(u, v)]dt + \frac{1}{\sqrt{\epsilon}}G_2(u, v)dW.$$

Where B_1 and B_2 are Lipschitz-Continuous.
In particular, our term $(\ , v^\epsilon)u^\epsilon$ or $(\ , v^\epsilon)\nabla u^\epsilon$ do not satisfy these assumptions.

If $u_0^\varepsilon \in L^2(D)$ and $v_0^\varepsilon \in L^2(\Omega; L^2(D))$ then there exists a unique global solution $u^\varepsilon \in L^\infty(\Omega; C([0,T]; L^2(D)) \cap L^2(0,T; H_0^1(D))))$ and $v^\varepsilon \in L^2(\Omega; C([0,T]; L^2(D)))$: P a.s.

$$\int_D u^\varepsilon(t)\varphi - \int_D u_0^\varepsilon \varphi + \int_0^t \int_D A^\varepsilon \nabla u^\varepsilon(s) \nabla \varphi + \int_0^t \int_D \rho^\varepsilon(v^\varepsilon) u^\varepsilon \varphi = \int_0^t \int_D f(s)\varphi ;$$

for every $t \in [0,T]$ and every $\varphi \in H_0^1(D)$, and

$$v^\varepsilon(t) = v_0^\varepsilon e^{-t/\varepsilon} + \frac{1}{\varepsilon} \int_0^t u^\varepsilon(s) e^{-(t-s)/\varepsilon} ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t e^{-(t-s}$$

Uniform estimates

$$\sup_{\varepsilon>0} \|u^\varepsilon\|_{L^\infty(\Omega;L^2(0,T;H_0^1(D)))} \leq C_T; \tag{11}$$

$$\sup_{\varepsilon>0} \|u^\varepsilon\|_{L^\infty(\Omega;C([0,T];L^2(D)))} \leq C_T; \tag{12}$$

and

$$\sup_{\varepsilon>0} \left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^\infty(\Omega;L^2(0,T;H^{-1}(D)))} \leq C_T. \tag{13}$$

$$\sup_{\varepsilon>0} \mathbb{E} \sup_{t\in[0,T]} \|v^\varepsilon(t)\|^2_{L^2(D)} \leq C_T. \tag{14}$$

The fast motion equation

For xed $2 L^2(D)$:

$$
\begin{aligned}
dv &= (v \quad)dt + {}^{p}\overline{Q}dW;\\
v(0) &= \ :
\end{aligned}
\qquad (15)
$$

This equation admits a unique mild solution
$v \ (t) \ 2 \ L^2(\ ; C(0; T; L^2(D)))$ given by:

$$
v \ (t) = \ e^{\ t} + \ (1 \quad e^{\ t}) + \int_0^t e^{\ (t \ s)} \, {}^{p}\overline{Q}dW: \qquad (16)
$$

The equation (15) admits a unique ergodic invariant measure that is strongly mixing and gaussian with mean and operator Q. We also have:

$$P_t \quad (\ ) \quad \int_{L^2(D)} \quad (z)d \quad (z) \quad c[\ ]e^{\ t}(1+k\ k_{L^2(D)}+k\ k_{L^2(D)});$$

for any Lipschitz function defined on $L^2(D)$, where [ ] is the Lipschitz constant of .

We need more refined results for the fast motion equation.

So for any $\;,\; \in L^2(\;, \mathcal{F}_{t_0}; L^2(D))$, and a.e. $\omega \in$ we have:

$$\mathbb{E}\left[\|v^{\;,}(t)\|^2_{L^2(D)} \,\middle|\, \mathcal{F}_{t_0} \right] \le 2\left(\|\;\|^2_{L^2(D)} e^{-2(t-t_0)} + \|\;\|^2_{L^2(D)} + Tr Q \right);$$

and

$$\mathbb{E}\left(P_t^{(\omega)}(\;(\omega)) - \int_{L^2(D)} (z)\, d^{(\omega)}(z) \,\middle|\, \mathcal{F}_{t_0} \right)$$

$$\le c[\;] e^{-(t-t_0)}(1 + \|\;(\omega)\|_{L^2(D)} + \|\;(\omega)\|_{L^2(D)});$$

Lemma

Let $\;\; 2\, C^u([0;T];L^1(\;\;;Lip(L^2(D))))$ *be an* F_t *- measurable process on* $Lip(L^2(D))$*, and let* $0 \quad t_0 < t_0 + \qquad T$*. For* $\;;\;\; 2\, L^2(\;\;;F_{t_0};L^2(D))$*, let* $v^{\;;}$ *be the previous solution. We have:*

$$\mathrm{E} \quad \frac{1}{\;} \int_{t_0}^{t_0+} \qquad (s;v^{\;;}(s)) \qquad \int_{L^2(D)} \qquad (s;z)d \quad (z) \; ! \quad ds \quad F_{t_0} \; !$$

$$c \quad 1 + k \quad k_{L^2(D)} + k \quad k_{L^2(D)} \quad \frac{k \quad k}{p\,\overline{\;}} + \sqrt{k \quad k[\quad](\;\;)} \quad ; \qquad (18)$$

where $[\quad]$ *is the modulus of uniform continuity of* $\quad$.

This lemma is crucial because, we need to apply the semigroup P_t to a function of the form

$$ {}^{"}(s;\ ) = \int_D {}^{"}(\ )u^{"}("s)\, dx $$

The cell problem

We introduce $: Y \to R$ the solution of the cell problem

$$\begin{aligned} \operatorname{div}\left(A(y)\left(I + r\ (y)\right)\right) &= 0 && \text{in } Y; \\ &Y\ \ \text{periodic}; \end{aligned}$$

Our main result of the di usion reaction equation

Our main result of the multicontinuum equation

Theorem (Bessaih-Efendiev-Maris, 2020)

Assume similar properties for initial conditions. Then, there exist $\bar u_1, \bar u_2 \in L^2(0,T;H_0^1(D)) \cap C([0,T];L^2(D))$ *such that* $u_1^\epsilon, u_2^\epsilon$ *converge in probability to* $\bar u_1, \bar u_2$*:*

$$\begin{cases} \dfrac{\partial \bar u_1}{\partial t} = \operatorname{div}\left(\bar A_1 \nabla \bar u_1\right) + \beta(g_1(\bar u_1,\bar u_2), g_2(\bar u_1,\bar u_2))(\bar u_2 - \bar u_1) + f_1 \text{ in } D, \\ \dfrac{\partial \bar u_2}{\partial t} = \operatorname{div}\left(\bar A_2 \nabla \bar u_2\right) + \beta(g_1(\bar u_1,\bar u_2), g_2(\bar u_1,\bar u_2))(\bar u_1 - \bar u_2) + f_2 \text{ in } D, \\ + \text{ initial conditions, boundary conditions}, \end{cases} \qquad (21)$$

Sketch of the proof

We need to pass to the limit in ε on the variational formulation

$$\int_D u^{\varepsilon}(t) \quad \int_D u_0^{\varepsilon} + \int_0^t \int_D A \ \nabla u^{\varepsilon}(s) \nabla + \int_0^t \int_D \varepsilon(v^{\varepsilon}) u^{\varepsilon}$$
$$= \int_0^t \int_D f(s) \ ;$$

Here, we use tightness arguments and pass to the limit in distribution only. After changing the space of probability, the sequence u^{ε} given by Skorokhod theorem converges a.s. to $\overline{u}$ strongly in $L^2(0; T; H_0^1(D))$

Key steps-for $S_1^{"}$

Fix $n^{"}$ a positive integer and let $\ ^{"} = \frac{T}{n^{"}}$. We de ne $\tilde{u}^{"}$ as the piecewise constant function:

$$\tilde{u}^{"}(t) = u^{"}(k\ ^{"}) \quad \text{for}$$

A simple calculation shows that

$$\lim_{\varepsilon \to 0} \|\tilde{u}^\varepsilon - u^\varepsilon\|_{L^\infty(0,T;L^2(D))} = 0; \qquad (23)$$

so we also have that

$$\lim_{\varepsilon \to 0} \|\tilde{v}^\varepsilon - v^\varepsilon\|_{L^\infty(0,T;L^2(D))} = 0; \qquad (24)$$

$$\int_0^T \int_D \left({}^{\varepsilon}(\tilde{v}^{\varepsilon}(t)) \quad \bar{}^{\varepsilon}(\tilde{u}^{\varepsilon}(t)) \right) \quad {}^{\varepsilon}(t)\,dx\,dt$$

$$= \sum_{k=0}^{n^{\varepsilon}-1} \int_{k\varepsilon}^{(k+1)\varepsilon} \int_D \left({}^{\varepsilon}(\tilde{v}^{\varepsilon}(t)) \quad \bar{}^{\varepsilon}(\tilde{u}^{\varepsilon}(t)) \right) \quad {}^{\varepsilon}(t)\,dx\,dt.$$

For the convergence of $S_3^{''}$, we used the homogenization results

- G. Allaire (1991), *Homogenization of the Navier-Stokes Equations with a Slip Boundary Condition.*

For any $t \in [0,T]$, $F_t^\varepsilon : L^2(D) \to L^2(D)$,

$$F_t^\varepsilon(z)(x) = \left(\frac{x}{\varepsilon}, z(x)\right) - \int_Y (y, z(x)) \quad u(t,x).$$

By a density argument, we show that: for any $z \in L^2(D)$, for every $t \in [0,T]$ and a.e. $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} F_t^\varepsilon(\ldots$$ [illegible]

The sequence being also uniformly bounded by $c\|\bar{u}\|_{L^\infty(\ ;C([0;T];L^2(D)))}\|\ \|_{L^\infty(D)}\|\ ^0\|_{L^\infty[0;T]}$. We apply the bounded convergence theorem and integrate over $\ \ [0;T]$ and get that

$$\lim_{\varepsilon \to 0} \mathrm{E}\,|S_3^\varepsilon| = 0.$$

The convergence of S_2^ε is simpler:

$$\mathrm{E}\,|S_2^\varepsilon| \quad c\|\ \ \|_{L^\infty(D)}\|\ \ \|_{L^\infty[0;T]}\mathrm{E}\,\|u^\varepsilon \quad \bar{u}\|_{L^1(0;T;H_0^1(D)))}.$$

implies that

$$\lim_{\varepsilon \to 0} \mathrm{E}\,|S_2^\varepsilon| = 0.$$

Combining the convergences of S_1^{ε}, S_2^{ε} and S_3^{ε} we get that

$$\lim_{\varepsilon \to 0} \mathbb{E} \int_0^T \int_D \left(\;{}^{\varepsilon}(v^{\varepsilon}(t))u^{\varepsilon}(t) \qquad \overline{}(\overline{u}(t))\overline{u}(t)\right) \; dxdt$$

Some remarks

- Tackle the full di usion problem
- Tackle the case of coe cient dependent on time, the non-autonomous case
- Generalize to the case of SPDEs for the particle equations
- Find some rate of convergence. This is related to better convergence, like convergence in mean.

References

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- H. Bessaih, Y. Efendiev, F. Maris, *Stochastic homogenization for a di usion-convection equation*, Discrete Contin. Dyn. Syst., **39** (2019), no. 9, 5403{5429.
- H. Bessaih, F. Maris, *Stochastic homogenization of multicontinuum heterogeneous ows*, Journal CAM, Vol **374** 2020.
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