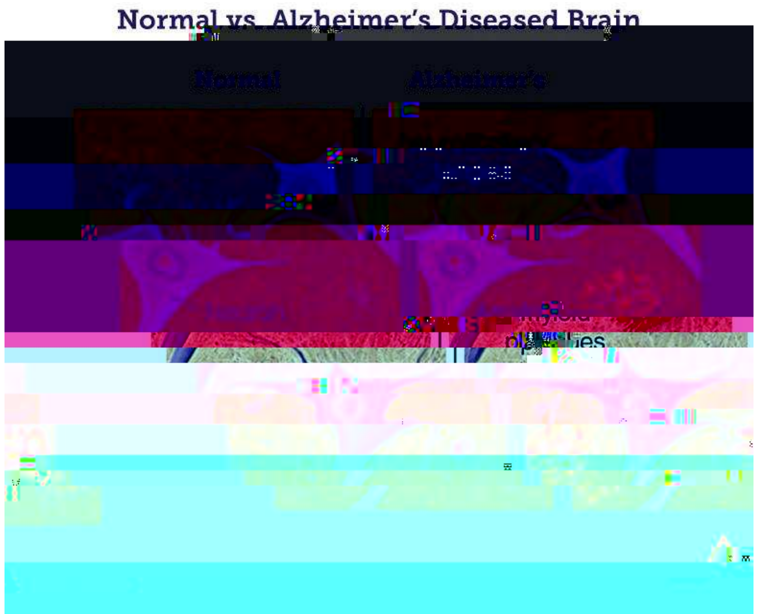
Alzheimer Disease (AD)



Alzheimer disease is characterized pathologically by the formation of senile plaques composed of **β**-amyloid peptide (A**β**). A**β** is naturally present in the brain and cerebrospinal fluid of humans throughout life. By unknown reasons (partially genetic), some neurons start to present an imbalance between production and clearance of A**β** amyloid during aging. Therefore, neuronal injury is the result of ordered A**β** self-association.

Under these assumptions, the discrete diffusive coagulation equations read

$$\frac{\partial u_i}{\partial t}(t,x - d_i \,\square_x u_i(t,x = Q_i(u \qquad \text{in } [0,T] \times Q, \tag{1}$$

where Q is the spatial domain and $[0,T]$ a time interval.

The variable $u_i(t,x) \geq 0$ [illegible]

A Mathematical Model for the Aggregation and Diffusion of β-Amyloid Peptide

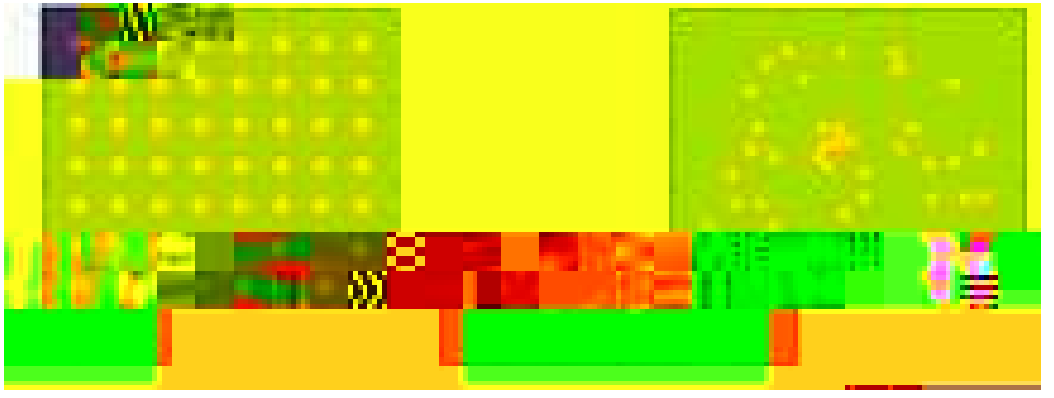


Figure 1: Periodically (left) and randomly (right) perforated domains.

In the present work, we account for the non-periodic cellular structure of the brain.

The distribution of neurons is modeled in the following way: there exists a family of predominantly genetic causes, not wholly deterministic, which influences the position of neurons and the microscopic structure of the parenchyma in a portion of the brain tissue Q.

We consider non-periodic random diffusion coefficients and a random production of Aβ in the monomeric form at the level of neuronal membranes.

This together defines a probability space $(\quad ,\mathbf{F},\mathbb{P}$.

Denoting by $\boldsymbol{\omega}$ □ the random variable in our model, the set of random holes in $\mathbb{R}^{n}$ (representing the neurons) is labeled by $\mathbf{G}(\boldsymbol{\omega}$.

The production of **β**-amyloid at the boundary $\boldsymbol{\Gamma(\omega)}$ of $\mathbf{G(\omega)}$

if $1 < s < M$

$$\frac{\partial u_s}{\partial t} - \operatorname{div}\left(D_s\left(t, x, \tau_{\frac{x}{\varepsilon}}\omega\right) \nabla_x u_s\right) + u_s \sum_{j=1}^{M} a_{s\,j} u_j = \frac{1}{2} \sum_{j=1}^{s-1} a_{j\,s-j}\, u_j\, u_{s-j} \quad \text{in } [0, T] \times Q$$

$$\left[D_s\left(t, x, \tau_{\frac{x}{\varepsilon}}\omega\right) \nabla_x u_s\right] \cdot n = 0 \quad \text{on } [0, T] \times \partial Q$$

$$\left[D_s\left(t, x, \tau_{\frac{x}{\varepsilon}}\omega\right) \nabla_x u_s\right] \cdot \nu_{\Gamma^{\varepsilon}_{Q}} = 0 \quad \text{on } [0, T] \times \Gamma^{\varepsilon}_{Q}$$

$$u_s(0, x$$

We assume that the movement of clusters results only from a diffusion process described by a stationary ergodic random matrix

$$\left(d^s_{i\,j}(t, x, \tau_{\frac{x}{\varepsilon}}\omega) \right)_{i\,j=1} =: D_s(t, x, \tau_{\frac{x}{\varepsilon}}\omega) \qquad 1 \leq s \leq M$$

where $(t, x) \in [0, T] \times Q$.

The production of β-amyloid peptide by the malfunctioning neurons is described imposing a non-homogeneous Neumann condition on the boundary of the holes, randomly selected within our domain.

To this end, we consider on Q a stationary ergodic random function

$$\eta : [0, T] \times \overline{Q} \times \quad \to [0, 1] \tag{8}$$

where the value '0' is assigned to 'healthy' neurons while all the other values in $]0, 1]$ indicate different degrees of malfunctioning.

Moreover, we assume that η is an increasing function of time, since once the neuron has become 'ill', it can no longer regain its original state of health.

Stationary ergodic dynamical systems

Definition 1 (Dynamical system). *Let (, F, ℙ be a probab ty space An*
m *d ens ona dyna ca syste s de ned as a fa y of easurab e b ect ve app n s* τ_x *: → x* □ ℝ *sat sfy n t e fo ow n cond t ons*

(t e roup property $\tau_0 = \mathbb{1}$ *𝟙 s t e dent ty app n (* $\tau_{x+y} = \tau_x \circ \tau_y$ □**x, y** □ ℝ

(t e app n s $\tau_x : \Omega \to \Omega$ *preserve t e easure* ℙ *on* Ω *e for every* **x** □ ℝ *and every* ℙ *easurab e set* **F** □ **F** *we ave* $\mathbb{P}(\tau_x F = \mathbb{P}(F$

(t e ap $T : \Omega \times \mathbb{R} \to \Omega$ (ω, x) □ $\tau_x \omega$ *s easurab e for t e standard* **σ** *a ebra on t e product space w ere on* ℝ *we ta e t e Bore* **σ** *a ebra(*

Definition 2 (Ergodicity). *A dyna ca syste s ca ed er od c f one of t e fo ow n equ va ent cond t ons s fu ed*

(ven a easurab e and nvar ant funct on **f** *n* Ω *t at s*

$$□x □ \mathbb{R} \quad f(\omega = f(\tau_x \omega$$

a ost everyw ere n Ω *t en*

$$f(\omega = const \quad for \quad \mathbb{P} - \text{a.e. } \omega □ ;$$

(f **F** □ **F** *s suc t at* $\tau_x F = F$ □**x** □ ℝ *t en* $\mathbb{P}(F = 0$ *or* $\mathbb{P}(F = 1$

Definition 3 (Stationarity)**.** *G ven a probab ty space* $(\; , \mathbf{F}, \mathbb{P}$ *a rea va ued process s a easurab e funct on* $\mathbf{f} : \mathbb{R} \times \; \to \mathbb{R}$ *e w say* $\mathbf{f}$ *s stat onary f t e d str but on of t e rando var ab e* $\mathbf{f}(\mathbf{y}, \cdot \; : \; \to \mathbb{R}$

Let $\mathbf{L}^p($ $(1 \leq \mathbf{p} < \infty)$ denote the space formed by (the equivalence classes of) measurable functions that are $\mathbb{P}$-integrable with exponent $\mathbf{p}$ and $\mathbf{L}$

One important property of random measures is the following generalization of the Birkhoff ergodic theorem

Concerning the random geometries, we make the assumptions listed below.

Definition 4. *An open set* $\mathbf{G} \square \mathbb{R}$ *s sa d to be n a y s oot w t constants* $(\boldsymbol{\delta}, \mathbf{N}, \mathbf{M}$ *f we ay cover* $= \boldsymbol{\partial}\mathbf{G}$ *by a countab e sequence of open sets* $(\mathbf{U}_i$ $_{i \quad \mathbb{N}}$ *suc t at*

- *Eac* $\mathbf{x} \square \mathbb{R}$ *s conta ned n at ost* $\mathbf{N}$ *of t e open sets* $\mathbf{U}$.
- *For any* $\mathbf{x} \square$ *t e ba* $\mathbf{B}_{\delta}(\mathbf{x})$ *s conta ned n at east one* $\mathbf{U}$.
- *For any* $\mathbf{i}$ *t e port on of t e boundary* $\mathbf{\Gamma}$ *ns de* $\mathbf{U}$ *a rees n so e Cartes an syste of coord nates, w t t e rap of a L psc tz funct on w ose L psc tz se nor s at ost* $\mathbf{M}$

Let Q be a bounded domain in $\mathbb{R}$. For given constants $(\boldsymbol{\delta}, \mathbf{N}, \mathbf{M}$, we consider $\mathbf{G}(\boldsymbol{\omega}$ a random open set which is a.s. minimally smooth with constants $(\boldsymbol{\delta}, \mathbf{N}, \mathbf{M}$ (uniformly minimally smooth).

We furthermore assume that $\mathbf{G}(\boldsymbol{\omega} := \square_{i \quad \mathbb{N}} \mathbf{G}_i(\boldsymbol{\omega}$ is a countable union of disjoint open balls $\mathbf{G}_i(\boldsymbol{\omega}$ with a maximal diameter $\mathbf{d}_0$.

We consider $\mathbf{G}\ (\boldsymbol{\omega}\ := \boldsymbol{\varepsilon}\mathbf{G}(\boldsymbol{\omega}$ and

□

$$Q\ (\boldsymbol{\omega}\ := Q\backslash$$

Stochastic two-scale convergence

Definition 5. *Let* $:= (\boldsymbol{\psi}_i \;\; i \;\; \mathbb{N}$ *be t e countab e dense fa y of* $\mathbf{C}_b(\Omega)$ *funct ons* $= (\boldsymbol{\phi}_i \;\; i \;\; \mathbb{N}$ *be a countab e dense subset of* $\mathbf{C}(\overline{Q}$ $\boldsymbol{\omega}$ □ Ψ *set of fu easure and* $\mathbf{u}$ □ $\mathbf{L}^2(0, \mathbf{T}; \mathbf{L}^2(Q$ *for a* $\boldsymbol{\varepsilon} > 0$

e say t at **u**

Domains with holes

Lemma 7. *Let* $\mathbf{u} \;\square\; \mathbf{L}^2(Q$ *be a sequence of funct ons suc t at* $\sup \;{}_0 \square\mathbf{u}\;\square_{L^2(Q)} < \infty$
If $(\mathbf{u}\,'\;,\;{}_0$ *s a subsequence suc t at* $\mathbf{u}\,' \overset{2s}{\square} \mathbf{u}$ *for so e* $\mathbf{u} \,\square\, \mathbf{L}^2(Q;\mathbf{L}^2($ *t en*
$\mathbf{u}\;\chi_{Q^\varepsilon} \overset{2s}{\square} \mathbf{u}\,\chi_{G^c}$

Lemma 8. *Let* $\mathbf{u} \;\square\; \mathbf{L}^2(0,\mathbf{T};\mathbf{H}^1(Q\;(\omega$ *be a sequence of funct ons suc t at*

$$\sup_{0} \square\mathbf{u}\;\square_{L^2(0\;\;;H^1(Q^\varepsilon(\;)))} + \square\partial_t\mathbf{u}\;\square_{L^2(0\;\;;L^2(Q^\varepsilon(\;)))} < \infty .$$

en t ere ex st funct ons $\mathbf{u} \,\square\, \mathbf{L}^2(0,\mathbf{T};\mathbf{H}^1(Q$ *w t* $\partial_t\mathbf{u} \,\square\, \mathbf{L}^2(0,\mathbf{T};\mathbf{L}^2(Q$ *and*
$\mathbf{v} \,\square\, \mathbf{L}^2(0,\mathbf{T};\mathbf{L}^2(Q;\mathbf{L}^2_{\mathrm{pot}}($ *suc t at* $\mathbf{E}\;\mathbf{u}\;\square\mathbf{u}$ *wea y n* $\mathbf{L}^2(0,\mathbf{T};\mathbf{H}^1(Q$ *and*
$\mathbf{E}\;\mathbf{u}\;\to\mathbf{u}$ *stron y n* $\mathbf{L}^2(0,\mathbf{T};\mathbf{L}^2(Q$ *as we as*

$$\mathbf{u} \overset{2s}{\square} \chi_{G^c}\,\mathbf{u}, \quad \partial_t\mathbf{u} \overset{2s}{\square} \chi_{G^c}\,\partial_t\mathbf{u}, \quad \text{and} \quad \square\mathbf{u} \overset{2s}{\square} \chi_{G^c}\square\mathbf{u} + \chi_{G^c}\,\mathbf{v}.$$

Homogenization

We obtain the following “deterministic” (i.e. for fixed ω □ Ω) existence and regularity result.

Theorem 2. *uppose a t e assu pt ons on our rando do a n o d*

en for $\mathbb{P}$ *a e* ω □ *and for any* $\varepsilon > 0$ *t e syste ad ts a un que ax a c ass ca so ut on*

$$\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_M$$

suc t at

t ere ex sts α □ $(0, 1$ α *depend n on y on*

In the sequel we shall rely on the fact that statements that hold $\mathbb{P}$-a.e. can be seen as deterministic assertions, since they hold whenever $\boldsymbol{Q}$

Main statement

Theorem 5. *Let* $\mathbf{u}_s(\mathbf{t},\mathbf{x}$ $1 \le \mathbf{s} \le \mathbf{M}$ *be a fa   y of nonne at ve c ass ca so ut ons to t e syste*

Denote by a t de t e extens on by zero outs de Q $(\boldsymbol{\omega}$ *and et* $\boldsymbol{\chi}_{G^{\complement}}$ *represent t e c aracter st c funct on of t e rando   set* $\mathbf{G}^{\complement}(\boldsymbol{\omega}$ *w ere* $\mathbf{G}^{\complement}$ *s t e co   p e   ent of* $\mathbf{G}$ *represent n   t e set of rando   o es n* $\mathbb{R}$

en t e sequences $(\widetilde{\mathbf{u}}_s$ $_0$ $(\widetilde{\Box_x \mathbf{u}_s}$ $_0$ *and* $(\widetilde{\partial_t \mathbf{u}_s}$ $_0$ $1 \le \mathbf{s} \le \mathbf{M}$ *stoc ast ca y two sca e conver e to* $[\boldsymbol{\chi}_{G^{\complement}}\, \mathbf{u}_s(\mathbf{t},\mathbf{x}\,]$ $[\boldsymbol{\chi}_{G^{\complement}}(\Box_x \mathbf{u}_s(\mathbf{t},\mathbf{x} + \mathbf{v}_s(\mathbf{t},\mathbf{x},\boldsymbol{\omega}\,]$ $[\boldsymbol{\chi}_{G^{\complement}}\, \partial_t\, \mathbf{u}_s(\mathbf{t},\mathbf{x}\,]$ $1 \le \mathbf{s} \le \mathbf{M}$ *respect ve y*

e   t n   funct ons $[(\mathbf{t},\mathbf{x}\ \mathrm{p}$

If $\mathbf{s} = 1$:

$$
\begin{cases}
\theta \dfrac{\partial \mathbf{u}_1}{\partial \mathbf{t}}(\mathbf{t},\mathbf{x}) - \mathrm{div}_x \left(\mathbf{D}_1(\mathbf{t},\mathbf{x}) \nabla_x \mathbf{u}_1(\mathbf{t},\mathbf{x}) \right) \\
+ \theta\, \mathbf{u}_1(\mathbf{t},\mathbf{x}) \left(\sum_{j=1}^{M} \mathbf{a}_{1\,j}\, \mathbf{u}_j(\mathbf{t},\mathbf{x}) \right) = \int_{\Omega} \chi_{\Gamma_{GC}}\, \eta(\mathbf{t},\mathbf{x},\omega)\, d\mu_{\Gamma_{\mathcal{P}}}(\omega) \quad \text{in } [0,\mathbf{T}] \times Q \\
[\mathbf{D}_1(\mathbf{t},\mathbf{x}) \nabla_x \mathbf{u}_1
\end{cases}
$$

If $s = M$:

$$\begin{cases} \theta \dfrac{\partial u_M}{\partial t}(t,x) - \operatorname{div}_x \left(D_M(t,x) \nabla_x u_M(t,x) \right) \\ = \dfrac{\theta}{2} \sum_{j+ \quad M} a_j \; u_j(t,x) \; u \; (t,x) & \text{in } [0,T] \times Q \\ \quad M(\text{if } j=M) \\ j \quad M(\text{if } \;=M) \\ [D_M(t,x) \nabla_x u_M(t,x)] \cdot n = 0 & \text{on } [0,T] \times \partial Q \\ u_M(0,x) = 0 & \text{in } Q \end{cases} \tag{26}$$

where $\theta = \int_\Omega \chi_{G^c} \, d\mu_{\mathcal{P}}(\omega) = \mathbb{P}(G^c)$.

$D_s(t,x)$ is a deterministic matrix, called "effective diffusivity":

$$(D_s)_{ij}(t,x) = \int_\Omega \chi_{G^c} D_s(t,x,\omega)(w_i(t,x,\omega) + \hat{e}_i) \cdot (w_j(t,x,\omega) + \hat{e}_j) \, d\mathbb{P}(\omega)$$

$(w_i)_{1 \quad i} \quad \in L^2([0,T] \times Q; L^2_{\text{pot}}(G^c))$ the family of solutions of the following microscopic problem:

$$\begin{cases} -\operatorname{div} \left[D_s(t,x,\omega)(w_i(t,x,\omega) + \hat{e}_i) \right] = 0 & \text{in } G^c \\ D_s(t,x,\omega) [w_i(t,x,\omega) + \hat{e}_i] \cdot \nu_{\Gamma_{G^c}} = 0 & \text{on } \quad {}_{G^c}. \end{cases} \tag{27}$$

$$v_s(t,x,\omega) = \sum_{i=1} w_i(t,x,\omega) \frac{\partial u_s}{\partial x_i}(t,x) \quad (1 \leq s \leq M).$$

Proof of the main Theorem.

In view of the previous Theorems, the sequences

$\widetilde{(\mathbf{u}_s}_0$, $\widetilde{(\Box_x \mathbf{u}_s}_0$ and $\widetilde{\left(\frac{\partial \mathbf{u}_s}{\partial \mathbf{t}}\right)}_0$ $(1 \leq \mathbf{s} \leq \mathbf{M})$ are bounded in $\mathbf{L}^2([0, \mathbf{T}] \times$

In the case when $\mathbf{s} = 1$, let us multiply the first equation of (5) by the test function $\boldsymbol{\varphi}$. Integrating, the divergence theorem yields

$$\int_0 \int_{Q^{\varepsilon}(\ )} \frac{\partial \mathbf{u}_1}{\partial \mathbf{t}} \boldsymbol{\varphi}(\mathbf{t}, \mathbf{x}, \boldsymbol{\omega}) \, \mathbf{dx}\, \mathbf{dt} + \int_0 \int_{Q^{\varepsilon}(\ )} \left(\mathbf{D}_1(\mathbf{t}, \mathbf{x}, \boldsymbol{\tau}_{\frac{\mathbf{x}}{\varepsilon}} \boldsymbol{\omega}) \nabla_x \mathbf{u}_1, \nabla \boldsymbol{\varphi} \right) \mathbf{dx}\, \mathbf{dt}$$

$$+ \int_0 \int_{Q^{\varepsilon}(\ )} \mathbf{u}_1 \sum_{j=1}^{M} \mathbf{a}_{1\,j} \mathbf{u}_j \boldsymbol{\varphi}(\mathbf{t}, \mathbf{x}, \boldsymbol{\omega}) \, \mathbf{dx}\, \mathbf{dt} = \boldsymbol{\varepsilon} \int_0 \int_{\Gamma^{\varepsilon}_{\mathbf{Q}}(\ )} \boldsymbol{\eta}(\mathbf{t}, \mathbf{x}, \boldsymbol{\tau}_{\frac{\mathbf{x}}{\varepsilon}} \boldsymbol{\omega}) \, \boldsymbol{\varphi}(\mathbf{t}, \mathbf{x}, \boldsymbol{\omega}) \, \mathbf{dH}$$ [illegible]

The term on the right-hand side of (28) follows from the lemma

Lemma 9. *Let* $(\mathbf{g}_i\ \ i\ \ \mathbb{N}$ *be a countab e fa y n* $\mathbf{L}$ $(Q\times\ ;\mathbf{L}\times\boldsymbol{\mu}_{\Gamma\ \mathcal{P}}$ *en t ere ex sts a set of fu easure* $_\Psi$ *suc t at for a ost every* $\boldsymbol{\omega}$ $_\Psi$ *every* $\mathbf{i}\ \mathbb{N}$ *every* $\boldsymbol{\psi}$ *and every* $\boldsymbol{\phi}\ \mathbf{C}_b(\overline{Q}$ *t e fo ow n o ds*

$$\lim_{0}\int_Q \mathbf{g}_i\ \mathbf{x},\boldsymbol{\tau}_{\frac{x}{\varepsilon}}\boldsymbol{\omega}\ \boldsymbol{\phi}(\mathbf{x}\ \boldsymbol{\psi}(\boldsymbol{\tau}_{\frac{x}{\varepsilon}}\boldsymbol{\omega}\ \mathrm{d}\boldsymbol{\mu}_{\Gamma(\ \ )}(\mathbf{x}\ = \int_Q\int_\Omega \mathbf{g}_i(\mathbf{x},\tilde{\boldsymbol{\omega}}\ \boldsymbol{\phi}(\mathbf{x}\ \boldsymbol{\psi}(\tilde{\boldsymbol{\omega}}\ \mathrm{d}\boldsymbol{\mu}_{\Gamma\ \mathcal{P}}(\tilde{\boldsymbol{\omega}}\ \mathrm{d}\mathbf{x}.$$

The last term on the left-hand side of (28) has been obtained b

An integration by parts shows that (28) can be put in the strong form associated with the following homogenized system:

$$-\mathbf{div}\left[\mathbf{D}_1(\mathbf{t},\mathbf{x},\boldsymbol{\omega})\left(\nabla_x \mathbf{u}_1(\mathbf{t},\mathbf{x}) + \mathbf{v}_1(\mathbf{t},\mathbf{x},\boldsymbol{\omega})\right)\right] = 0 \qquad \text{in } [0,\mathbf{T}] \times Q \times \mathbf{G}^{\complement} \tag{29}$$

$$\left[\mathbf{D}_1(\mathbf{t},\mathbf{x},\boldsymbol{\omega})\left(\nabla_x \mathbf{u}_1(\mathbf{t},\mathbf{x}) + \mathbf{v}_1(\mathbf{t},\mathbf{x},\boldsymbol{\omega})\right)\right]\cdot \boldsymbol{\nu}_{\Gamma_{G^{\complement}}} = 0 \qquad \text{on } [0,\mathbf{T}] \times Q \times \partial G^{\complement} \tag{30}$$

$$\begin{aligned} &\boldsymbol{\theta}\,\frac{\partial \mathbf{u}_1}{\partial \mathbf{t}}(\mathbf{t},\mathbf{x}) - \mathbf{div}_x\left(\int_\Omega \boldsymbol{\chi}_{G^{\complement}}\,\mathbf{D}_1(\mathbf{t},\mathbf{x},\boldsymbol{\omega})\left(\nabla_x \mathbf{u}_1(\mathbf{t},\mathbf{x}) + \mathbf{v}_1(\mathbf{t},\mathbf{x},\boldsymbol{\omega})\right) d\mathbb{P}(\boldsymbol{\omega})\right) \\ &+ \boldsymbol{\theta}\,\mathbf{u}_1(\mathbf{t},\mathbf{x}) \sum_{j=1}^{M} \mathbf{a}_{1\,j}\,\mathbf{u}_j(\mathbf{t},\mathbf{x}) - \int_\Omega \boldsymbol{\chi}_{\Gamma_{G^{\complement}}}\,\boldsymbol{\eta}(\mathbf{t},\mathbf{x},\boldsymbol{\omega})\, d\boldsymbol{\mu}_{\Gamma,\mathcal{P}}(\boldsymbol{\omega}) = 0 \qquad \text{in } [0,\mathbf{T}] \times Q \end{aligned} \tag{31}$$

$$\left(\int_\Omega \boldsymbol{\chi}_{G^{\complement}}\,\mathbf{D}_1(\mathbf{t},\mathbf{x},\boldsymbol{\omega})\left(\nabla_x \mathbf{u}_1(\mathbf{t},\mathbf{x}) + \mathbf{v}_1(\mathbf{t},\mathbf{x},\boldsymbol{\omega})\right) d\mathbb{P}(\boldsymbol{\omega})\right)\cdot \mathbf{n} = 0 \qquad \text{on } [0,\mathbf{T}] \times \partial Q. \tag{32}$$

By using the relation (33) in Eqs.(31) and (32), we get

$$\theta \frac{\partial \mathbf{u}_1}{\partial \mathbf{t}}(\mathbf{t},\mathbf{x}) - \mathbf{div}_x\left(\mathbf{D}_1(\mathbf{t},\mathbf{x})\nabla_x \mathbf{u}_1(\mathbf{t},\mathbf{x})\right) + \theta\, \mathbf{u}_1(\mathbf{t},\mathbf{x}) \sum_{j=1}^{M} \mathbf{a}_{1\,j}\, \mathbf{u}_j(\mathbf{t},\mathbf{x}) - \int_{\Omega} \chi_{\Gamma_{G^c}}\, \eta(\mathbf{t},\mathbf{x},\omega)\, d\mu_{\Gamma\,\mathcal{P}}(\omega) = 0 \quad \text{in } [0,\mathbf{T}] \times Q \tag{35}$$

$$\left[\mathbf{D}_1 \nabla_x \mathbf{u}_1(\mathbf{t},\mathbf{x})\right] \cdot \mathbf{n} = 0 \qquad \text{on } [0,\mathbf{T}] \times \partial Q \tag{36}$$

where the entries of the matrix $\mathbf{D}_1$ (called ”effective diffusivity”) are given by

$$(\mathbf{D}_1)_{ij}(\mathbf{t},\mathbf{x}) = \int_{\Omega} \chi_{G^c}\, \mathbf{D}_1(\mathbf{t},\mathbf{x},\omega)\, [\mathbf{w}_i(\mathbf{t},\mathbf{x},\omega) + \hat{\ }$$

References

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