Markovian limits of the generalized McKean-Vlasov dynamics

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Karlstad Applied Analysis Seminar, 02 June 2021

(Weakly) interacting particle systems

1. The overdamped McKean-Vlasov dynamics (oMK-V)

$$\dot Q_i = -\nabla V(Q_i) - \frac{1}{N}\sum_{j=1}^{N} \nabla_q U(Q_i - Q_j) + \sqrt{2\beta^{-1}}\,\dot W_i,$$

V: confining potential, U: interaction potential

2. The underdamped McKean-Vlasov dynamics (uMK-V)

$$\ddot Q_i = -\nabla V(Q_i) - \frac{1}{N}\sum_{j=1}^{N} \nabla_q U(Q_i - Q_j) - \gamma \dot Q_i + \sqrt{2\gamma\beta^{-1}}\,\dot W_i,$$

3. The generalized McKean-Vlasov dynamics (gMK-V)

$$\ddot Q_i = -\nabla V(Q_i) - \frac{1}{N}\sum_{j=1}^{N} \nabla_q U(Q_i - Q_j) - \sum_{j=1}^{N}\int_0^t \gamma_{ij}(t-s)\,\dot Q_j(s)\,ds + F_i(t),$$

where $F(t) = (F_1(t), \ldots, F_N(t))$ is a mean zero, Gaussian, stationary process, and $[\gamma_{ij}(t-s)]_{i,j=1,\ldots,m}$ are autocorrelation functions.

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- Applications: statistical physics, mathematical biology, mathematical models in the social sciences (cooperative behavior, risk management and opinion formation)
- Many challenging mathematical problems:
    - Mean-field limits ($N \to \infty$)
    - Long-time behaviour ($t \to \infty$)
    - Phase transition (strength of the noise varies)
    - Model reduction (coarse-graining)
    - Hilbert's sixth problem

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Markovian approximation of the gMK-V dynamics

- Markovian approximation of the gMK-V dynamics,

$$
\begin{aligned}
dQ_i(t) &= P_i(t)\,dt\\
dP_i(t) &= -\nabla V(Q_i(t))\,dt - \frac{1}{N}\sum_{j=1}^{N} \nabla_q U(Q_i(t) - Q_j(t))\,dt + \lambda^T Z_i(t)\,dt\\
dZ_i(t) &= -\lambda P_i(t)\,dt - AZ_i(t) + \sqrt{2\beta^{-1}A}\,dW_i(t).
\end{aligned}
$$

- e.g., when approximating the memory kernel by a sum of exponentials

$$
\gamma_m(t) = \sum_{i=1}^{m} \lambda_i^2 e^{-\alpha_i |t|},
$$

one can take $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_m)$

[Kupferman, J. Stat. Phys, 2004.]

Mean-field limits ($N \to +\infty$)

- Each particle convergence to the following SDE (propagation of chaos)

$$
\begin{aligned}
dQ(t) &= P(t)\,dt; \\
dP(t) &= -\nabla V(Q(t))\,dt - \nabla_q U * \mu_t(Q_t)\,dt + \lambda^T Z(t)\,dt; \\
dZ(t) &= -\lambda P(t)\,dt - AZ(t) + \sqrt{2\beta^{-1}}A\,dW(t).
\end{aligned}
$$

- The empirical measure converges to the solution of

$$
\begin{aligned}
\partial_t \mu &= -\mathrm{div}_q(p\mu) + \mathrm{div}_p\Big[\big(\nabla_q V(q) + \nabla_q U * \mu(q) - \lambda^T z\big)\mu\Big] \\
&\quad + \mathrm{div}_z\Big[(\lambda p + Az)\mu\Big] + \beta^{-1}\mathrm{div}_z(A\nabla_z \mu);
\end{aligned}
$$

which is the forward Kolmogorov equation associated to the SDE, $\mu_t = \mathrm{Law}(Q_t; P_t; Z_t)$.

[**D.**, Nonlinear Analysis, 2015] using the coupling method. See [Golse, Lecture Notes in Applied Mathematics and Mechanics (Editors: Muntean, Rademacher & Zagaris), 2016] for a survey of the topics.

From the uMK-V dynamics to the oMK-V dynamics

Classical result: the oMK-V dynamics can be obtained from the uMK-V dynamics under the high-friction limit ($\gamma \to +\infty$) or the zero-mass limit ($m \to 0$).

Heuristic idea: the underdamped Langevin dynamics

$$m\ddot{Q} = -\nabla V(Q) - \gamma \dot{Q} + \sqrt{2\gamma\beta^{-1}}\dot{W}.$$

Sending $m \to 0$ yields the overdamped Langevin dynamics

$$\gamma\dot{Q} = -\nabla V(Q) + \sqrt{2\gamma\beta^{-1}}\dot{W}$$

Many different approaches, [D.-Lamacz-Peletier-Sharma, CVPDEs 2017] and [D.-Lamacz-Peletier-Schlichting-Sharma, Nonlinearity 2018]: variational approach and quantification of errors.

From the gMK-V dynamics to the uMK-V dynamics

- White noise (Markovian) limits: $\;\mapsto\;\;/\varepsilon$ and $A \mapsto A/\varepsilon^2$

Main steps of the proof

Step 1. The forward Kolmogorov (Fokker-Planck) equation

$$
\begin{aligned}
\frac{\partial\ \varepsilon}{\partial t} &= L \quad \varepsilon \\
&= \quad p \quad \nabla_q \quad \varepsilon + (\nabla V(q) + \nabla_q U(q) \quad {}^{\varepsilon}_{t}) \quad \nabla_p \quad \varepsilon \\
&\quad + \frac{1}{\varepsilon} \qquad {}^{T} z \quad \nabla_p \quad \varepsilon + p \qquad \nabla_z \quad \varepsilon \\
&\qquad + \frac{1}{\varepsilon^2} \ \mathrm{div}_p(Az \quad \varepsilon) + \qquad {}^{1}\mathrm{div}(A \nabla_p \quad \varepsilon) \\
&:= \quad L_2 + \frac{1}{\varepsilon} L_1 + \frac{1}{\quad}
\end{aligned}
$$

(iaεntkkmeasureSTd[(TJ/F89Tf6Td0)

Main steps of the proof (cont.)

Step 2: We define the function $f^{\varepsilon}(q;p;z;t)$ through

$$^{\varepsilon}(q;p;z;t) = \quad _1(q;p;z) f^{\varepsilon}(q;p;z;t).$$

The function $f^{\varepsilon}(q;p;z;t)$ satisfies the equation

$$\begin{aligned}
\frac{\partial f^{\varepsilon}}{\partial t} &= \quad p \quad \nabla_q f^{\varepsilon} + (\nabla V(q) + \nabla_q U(q) \quad (f^{\varepsilon} \quad _1)) \quad \nabla_p f^{\varepsilon} \\
&\quad + \quad f^{\varepsilon} p \quad \nabla U(q) \quad\quad _1(1 \quad f^{\varepsilon}) + \frac{1}{\varepsilon} \quad\quad ^T z \quad \nabla_p f^{\varepsilon} + p \quad\quad \nabla_z f^{\varepsilon} \\
&\quad + \frac{1}{\varepsilon^2} \quad\quad Az \quad \nabla_z f^{\varepsilon} + \quad\quad ^1 \mathrm{div}_z(A \nabla_z f^{\varepsilon}) \\
&=: \quad \hat{L}_2 + \frac{1}{\varepsilon}\hat{L}_1 + \frac{1}{\varepsilon^2}\hat{L}_0 \quad f^{\varepsilon}.
\end{aligned}$$

Main steps of the proof (cont.)

Step 3: We look for a solution of the form

$$f^{\varepsilon} = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots$$

We obtain the following sequence of equations

$$\begin{aligned} \hat{\mathcal{L}}_0 f_0 &= 0; \\ \hat{\mathcal{L}}_0 f_1 &= \hat{\mathcal{L}}_1 f_0; \\ \hat{\mathcal{L}}_0 f_2 &= \hat{\mathcal{L}}_2 f_0 + \hat{\mathcal{L}}_1 f_1 - \frac{\partial f_0}{\partial t} \end{aligned}$$

Step 4:

f_0 is independent of z:

$$f_0 = f(q;p;t):$$

The second equation becomes

$$\hat{L}_0 f_1 = \quad {}^T z \quad r_p f:$$

The solvability condition is satis ed [since $\;{}^T z \quad r_p f$ is orthogonal to the null space of $\hat{L}_0$ which consists of functions of the form $e^{\quad \frac{kzk^2}{2}} u(q;p)$]. Therefore, it has a unique solution, up to a term in the null space of $\hat{L}_0$,

$$f_1 = \quad zA^{\quad 1} \quad r_p f \quad (\text{thus } \hat{L}_1 f_1 = \quad {}^T A^{\quad 1} \quad kzk^2 \quad {}_p f \quad p \quad r_p f \quad):$$

The solvability condition for the last equation:

$$\int \quad \hat{L}_2 f + \hat{L}_1 f_1 \quad \frac{@f}{@t} \quad Z^{\quad 1} e^{\quad \frac{kzk^2}{2}} dz = 0:$$

Since $\hat{L}_2 f \quad \frac{@f}{@t}$ does not depends on z,

$$\frac{@f}{@t} = \hat{L}_2 f + \int (\hat{L}_1 f_1) Z^{\quad 1} e^{\quad \frac{kzk^2}{2}} dz:$$

Direct computations give

$$\hat{L}_2 f = p \; r_q f + (r V(q) + r_q U(q) \; (f \hat{\ }_1)) \; r_p f + \; fp \; r U(q) \; \hat{\ }_1 (1 \; f);$$

where $\hat{\ }_1$ satis es

$$\hat{\ }_1 (q;p) = \exp \Big[\; \frac{|p|^2}{2} + V(q) + U \; \hat{\ }_1 (q);$$

Summary

Interacting particle systems:

- mean-field limit
- phase-transition
- model reduction

Future work

- singular interactions
- non-Markovian systems

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