Relative energy approach to a diffuse interface model of a compressible two-phase flow

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Outline of the talk

1. Introduction of the model
2. Dissipative weak solutions
3. Relative energy inequality
4. Applications: weak-strong uniqueness principle
5. Applications: low Mach number limit
6. Applications: some numerics on the model

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Introduction of the model

Overview on the existing results

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Introduction of the model

Aim: describe the dynamics of a binary mixture of compressible, viscous and macroscopically immiscible fluids in a bounded domain

Possible approaches:

1.Classical approach: the interface between the two fluids is sharp and is evolving in time along with the fluid, the movement of the interface at each time is determined by a set of interfaci]TJ0 0 7Aalancico

Illustration of a diffuse interface versus a sharp interface

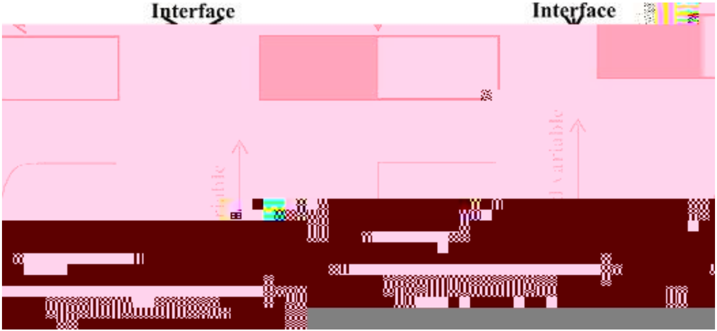

Figure: Illustration of a diffuse interface and of a sharp interface for a mixture of two immiscible fluids

Phase-field approach

- Interface of finite width "
- c is an order parameter (e.g. the concentration difference) that takes value 1 in the bulk of one fluid and value -1 in the bulk of the other fluid, varying continuously between -1 and 1 at the interface
- c satisfies an equation of Allen-Cahn type
- the fluid density and respectively velocity $(\rho, \boldsymbol{v})$ satisfy the compressible Navier-Stokes equations coupled to the Allen-Cahn equation through a capillarity force ($\mathrm{div}_x \quad r_x c \quad r_x c - \frac{1}{2}|r_x c|^2 \mathrm{I}$

The compressible Navier-Stokes-Allen-Cahn Model

The Blesgen model:

$$\partial_t \rho + \mathrm{div}_x(\rho \mathbf{u}) = 0, \tag{1}$$

$$\partial_t(\rho \mathbf{u}) + \mathrm{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\rho, c) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) - \mathrm{div}_x\left(\nabla_x c \otimes \nabla_x c - \frac{1}{2}|\nabla_x c|^2 \mathbb{I}\right), \tag{2}$$

$$\partial_t(\rho c) + \mathrm{div}_x(\rho c \mathbf{u}) = -\mu, \tag{3}$$

$$\rho\mu = -\Delta_x c + \rho \frac{\partial f(\rho, c)}{\partial c}. \tag{4}$$

The tensor $\mathbb{S}(\nabla_x \mathbf{u}) = \nu\left(\nabla_x \mathbf{u}^t + \nabla_x^t \mathbf{u} - \frac{2}{N}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \lambda \mathrm{div}_x \mathbf{u}\mathbb{I}$, $\nu > 0$.
The free energy is $E_{\mathrm{free}}(\rho, c, \nabla_x c) = \frac{1}{2}|\nabla_x c|^2 + \rho f(\rho, c)$.
The pressure is derived from the free energy $p(\rho, c) = \rho^2 \frac{\partial f(\rho, c)}{\partial \rho}$.

Overview on the existing results

Theoretical study of the model: existence of weak and strong solutions, presence of initial vacuum

Blesgen (1999): introduction of the model

Feireisl, Petzeltov´

Overview on the existing results

A simplified version

We consider the free energy in a simplified form

$$E_{\text{free}} = \frac{1}{2}|\nabla_x c|^2 + F_c(c) + F_e(\rho), \tag{5}$$

yielding the pressure

$$p(\rho, c) = p_e(\rho) - F_c(c), \; p_e(\rho) = \rho F_e'(\rho) - F_e(\rho)\,. \tag{6}$$

The Allen-Cahn equation taken in a simplified form:

$$\partial_t c + \mathbf{u} \cdot \nabla_x c = \Delta_x c - F_c'(c). \tag{7}$$

Boundary conditions: $\mathbf{u} = 0$, $c = c_b$ on $\partial\Omega$.

$$\begin{aligned} & p_e \in C[0,\infty) \cap C^1(0,\infty), \\ & p_e'(\rho) > 0 \text{ for } \rho > 0, \; \liminf_{\rho \to \infty} p_e'(\rho) > 0, \; p_e(\rho) \le c\,(1 + F_e(\rho)) \text{ for all } \rho > 0, \end{aligned} \tag{8}$$

$$F_c \in C^1(R), \; F_c''(c) > 0 \text{ for all } c \in (-\infty, -\overline{c}] \cup [\overline{c}, \infty), \; \overline{c} > 0. \tag{9}$$

Typically $F_e(\rho) = a\rho^{\gamma}$, $a > 0$, $\gamma > 1$, $F_c = (c^2 - 1)^2$ regular double well potential.

Dissipative weak solutions

Let the initial data,

$$\rho(0,\cdot) = \rho_0, \ \rho\mathbf{u}(0,\cdot) = (\rho\mathbf{u})_0, \ c(0,\cdot) = c_0, \tag{10}$$

be given in the class

$$\rho_0 > 0 \text{ a.e. in } \Omega, \ \int_\Omega \left(\frac{|(\rho\mathbf{u})_0|^2}{\rho_0} + F_e(\rho_0) \right) dx < \infty, \tag{11}$$
$$c_0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega), \ c_0|_{\partial\Omega} = c_b.$$

We search for $[\rho, \mathbf{u}, c]$ in the class of functions

$$\rho, F_e(\rho) \in L^\infty(0,T;L^1(\Omega)), \ \rho > 0 \text{ a.a. in } (0,T)\times\Omega,$$
$$\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;R^N)),$$
$$c \in L^\infty(0,T;W^{1,2}(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)) \cap L^\infty((0,T)\times\Omega), \ c|_{\partial\Omega} = c_b;$$

Dissipative weak solutions

- The integral identity for ρ

$$\left[\int_\Omega \rho \varphi \, dx\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\rho \partial_t \varphi + \rho \mathbf{u} \cdot \nabla_x \varphi\right] dx\, dt, \ \forall \tau > 0, \ \varphi \in C^1_c([0,T] \times \overline{\Omega});$$

- The integral identity for the momentum equation

$$\left[\int_\Omega \rho \mathbf{u} \cdot \boldsymbol{\varphi} \, dx\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \left[\rho \mathbf{u} \partial_t \boldsymbol{\varphi} + \rho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p_e(\rho) \mathrm{div}_x \boldsymbol{\varphi}\right] dx\, dt$$
$$- \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} \, dx\, dt$$
$$+ \int_0^\tau \int_\Omega \nabla_x c \otimes \nabla_x c - \frac{1}{2} |\nabla_x c|^{\dots}$$

Dissipative weak solutions

- The Allen-Cahn equation:

$$\partial_t c + \mathbf{u}\cdot\nabla_x c = \Delta_x c - F_c'(c),\ \text{a.a. } (0,T)\times\Omega,\ c(0,\cdot) = c_0,\ c|_{\partial\Omega} = c_b;$$

- The energy inequality:

$$\left[\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\nabla_x c|^2 + F_e(\rho) + F_c(c)\right)\mathrm{d}x\right]_{t=0}^{t=\tau}$$
$$+\int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega [\Delta_x c - F_c'(c)]^2\,\mathrm{d}x\,\mathrm{d}t \leq 0,\ \forall\tau > 0.$$

Weak-strong uniqueness

Theorem

Let $\Omega \quad R^N$, $N = 1, 2, 3$ *be a bounded Lipschitz domain. Let* $[\rho, \mathbf{u}, c]$ *be a dissipative weak solution in* $(0, T) \times \Omega$ *in the sense specified above. Suppose that the same problem admits a classical solution* $[r, \mathbf{U}, C]$, $r > 0$ *defined on the same time interval. Then*

$$\rho = r, \; \mathbf{u} = \mathbf{U}, \; c = C \text{ in } (0, T) \times \Omega.$$

Low Mach number limit

After rescaling the elastic pressure:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p_e(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$- \mathrm{div}_x \quad \nabla_x c \otimes \nabla_x c - \frac{1}{2}|\nabla_x \mathrm{u}$$

Low Mach number limit

Theorem

Suppose that problem (14) *admits a smooth solution* $[\mathbf{U}, C]$, *with the initial data* $[\mathbf{U}_0, C_0]$, *on a time interval* $[0, T]$. *Suppose*

$$\rho(0,\cdot) = \rho_{0,\varepsilon} = 1 + \varepsilon \rho_{0,\varepsilon}^{(1)}, \quad \int_\Omega \rho_{0,\varepsilon}^{(1)} \, dx = 0,$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \ \rho_{0,\varepsilon}^{(1)} \to 0 \text{ in } L^1(\Omega), \ \mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ in } L^2(\Omega; R^N),$$

$$c(0,\cdot) = c_{0,\varepsilon} \in L^1 \cap W_0^{1,2}(\Omega), \ \|c_{0,\varepsilon}\|_{L^1(\Omega)} \lesssim 1, \ c_{0,\varepsilon} \to C_0 \text{ in } W_0^{1,2}(\Omega)$$

as $\varepsilon \to 0$. *Let* $[\rho_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon]_{\varepsilon>0}$ *be a dissipative weak solution of the compressible NSAC with the initial data* $[\rho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}, c_\varepsilon]$, $\|c_\varepsilon\|_{L^1((0,T)\times\Omega)} \lesssim 1$. *Then*

$$\rho_\varepsilon(t,\cdot) \to 1 \text{ in } L^1(\Omega), \ \mathbf{u}_\varepsilon(t,\cdot) \to \mathbf{U}(t,\cdot) \text{ in } L^2(\Omega; R^N), \ c_\varepsilon(t,\cdot) \to C(t,\cdot) \text{ in } W^{1,2}(\Omega)$$

uniformly for $t \in [0, T]$.

Relative energy inequality

$$
\Big[\mathcal{E}\big(\rho, c, \mathbf{u} \,\big|\, r, C, \mathbf{U}\big) \Big]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \Big[\mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) + \mu^2 \Big] \, dx \, dt
$$

$$
\leq - \int_0^\tau \int_\Omega \big(F_c(c) \mathrm{div}_x \mathbf{u} + \mu F_c'(c) \big) \, dx \, dt
$$

$$
+ \int_0^\tau \int_\Omega \big(\rho (\mathbf{U} - \mathbf{u}) \cdot \partial_t \mathbf{U} + \rho \mathbf{u} \cdot (\mathbf{U} - \mathbf{u}) \cdot \nabla_x^t \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) \big) \, dx \, dt
$$

$$
- \int_0^\tau \int_\Omega \big(\rho \partial_t F_e'(r) + \rho \mathbf{u} \cdot \nabla_x F_e'(r) \big) \, dx \, dt - \int_0^\tau \int_\Omega p_e(\rho) \mathrm{div}_x \mathbf{U} \, dx \, dt
$$

$$
- \int_0^\tau \int_\Omega \Big(\nabla_x c \otimes \nabla_x c - \frac{1}{2} |\nabla_x c|^2 \mathbb{I} \Big) : \nabla_x \mathbf{U} \, dx \, dt + \int_0^\tau \int_\Omega F_c(c) \mathrm{div}_x \mathbf{U} \, dx \, dt
$$

$$
+ \int_0^\tau \int_\Omega \partial_t (\Delta_x C) c \, dx \, dt + \int_0^\tau \int_\Omega \Delta_x C \, [\mu - \mathbf{u} \cdot \nabla_x c] \, dx \, dt
$$

$$
+ \int_0^\tau \int_\Omega \partial_t \int_\Omega +
$$

Weak-strong uniqueness

Idea: Take $[r, \mathbf{U}, C]$ a strong solution of the problem, use it as test function in the relative energy inequality and apply a Gronwall type argument.

Convective term in the equation of continuity:

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \left(\rho(\mathbf{U}-\mathbf{u})\cdot\partial_t \mathbf{U} + \rho\mathbf{u}\cdot\nabla_x^t \mathbf{U}\cdot(\mathbf{U}-\mathbf{u}) \right) \mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_\Omega \left(\rho(\mathbf{U}-\mathbf{u})\cdot\partial_t \mathbf{U} + \rho(\mathbf{U}-\mathbf{u})\cdot\nabla_x^t \mathbf{U}\cdot\mathbf{U} \right) \mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \rho(\mathbf{u}-\mathbf{U})\cdot\nabla_x^t \mathbf{U}\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_0^\tau \int_\Omega \left[\left(\frac{\rho}{r}-1\right)(\mathbf{U}-\mathbf{u})\cdot\left(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C\Delta_x C + \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})\right) \right] \mathrm{d}x\,\mathrm{d}t \\
&+ c_1 \int_0^\tau \mathcal{E}\left(\rho, c, \mathbf{u} \,\middle|\, r, C, \mathbf{U}\right) \mathrm{d}t + \int_0^\tau \int_\Omega \left[(\mathbf{U}-\mathbf{u})\cdot\left(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C\Delta_x C\right) \right] \mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau \int_\Omega (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) : \mathbb{S}(\nabla_x \mathbf{U})\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

We can prove that:

$$\int_0^\tau\int_\Omega \left[\left(\frac{\rho}{r} - 1\right) (\mathbf{U} - \mathbf{u}) \cdot \left(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C \Delta_x C + \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \right) \right] \mathrm{d}x\, \mathrm{d}t$$
$$\leq c_2 \int_0^\tau \mathcal{E}\left(\rho, c, \mathbf{u} \,\middle|\, r, C, \mathbf{U}\right) \mathrm{d}t + \frac{1}{2}\int_0^\tau\int_\Omega \left(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})\right) : \left(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}\right) \mathrm{d}x\, \mathrm{d}t.$$

Technical ingredient: We introduce a cut-off function $\Psi \in C_c^1(0, \infty)$, $0 \leq \Psi \leq 1$, $\Psi \equiv 1$ in $[\delta, \frac{1}{\delta}]$, where δ s.t. $r(t,x) \in [2\delta, \frac{1}{2\delta}]$ $\forall\,(t,x) \in [0,T] \times \overline{\Omega}$.
For $h \in L^1((0,T) \times \Omega)$, we set
$h = h_{\mathrm{ess}} + h_{\mathrm{res}}$, $h_{\mathrm{ess}} = \Psi(\rho) h$, $h_{\mathrm{res}} = (1 - \Psi(\rho)) h$. We can prove:

$$F_e(\rho) - F_e'(r)(\rho - r) - F_e(r) \lesssim (\rho - r)^2_{\mathrm{ess}} + (1 + \rho)_{\mathrm{res}}$$

and

$$\mathcal{E}\left(\rho, \mathbf{u}, c \,\middle|\, r, \mathbf{U}, C\right) \gtrsim \int_\Omega \left(\rho|\mathbf{u} - \mathbf{U}|^2 + [\rho - r]^2_{\mathrm{ess}} + 1_{\mathrm{res}} + \rho_{\mathrm{res}} \right) \mathrm{d}x.$$

Thus,

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \left[\left(\frac{\rho}{r} - 1\right) (\mathbf{U} - \mathbf{u}) \cdot \left(\nabla_x F_c(C) - \nabla_x p_e(r) - \nabla_x C \Delta_x C + \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \right) \right] \, \mathrm{d}x \, \mathrm{d}t \\
&\lesssim \int_0^\tau \int_\Omega |\rho - r| \, |\mathbf{U} - \mathbf{u}| \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_0^\tau \int_\Omega |[\rho - r]_{\mathrm{ess}}| \, |\mathbf{U} - \mathbf{u}| \, \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \int_\Omega |[\rho - r]_{\mathrm{res}}| \, |\mathbf{U} - \mathbf{u}| \, \mathrm{d}x \, \mathrm{d}t \\
&\leq c(\delta) \int_0^\tau \int_\Omega \left([\rho - r]_{\mathrm{ess}}^2 + 1_{\mathrm{res}} + \rho_{\mathrm{res}} + \rho |\mathbf{u} - \mathbf{U}|^2 \right) \mathrm{d}x \, \mathrm{d}t + \delta \int_0^\tau \int_\Omega |\mathbf{u} - \mathbf{U}|^2 \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}
$$

Using the Korn–Poincaré inequality

$$
\int_\Omega \left(|\mathbf{u} - \mathbf{U}|^2 + |\nabla_x (\mathbf{u} - \mathbf{U})|^2 \right) \mathrm{d}x \leq c_{kp} \int_\Omega \left(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) \right) : \left(\nabla_x \mathbf{u} - \nabla_x \mathbf{U} \right) \mathrm{d}x,
$$

we get the desired inequality.

Terms containing the order parameter:

$$
\begin{aligned}
&\Big[\mathcal{E}\big(\rho, c, \mathbf{u} \,\big|\, r, C, \mathbf{U}\big)\Big]_{t=0}^{t=\tau} + \int_0^\tau\int_\Omega \frac{1}{4}\big(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})\big) : \big(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}\big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau\int_\Omega |\Delta_x c - \Delta_x C|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_0^\tau\int_\Omega \Delta_x C(\nabla_x C - \nabla_x c)\cdot(\mathbf{u} - \mathbf{U})\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau\int_\Omega \nabla_x \mathbf{U} : \Big(\nabla_x(C - c) \otimes \nabla_x(C - c) - \frac{1}{2}|\nabla_x(C - c)|^2 \mathbb{I}\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau\int_\Omega \mathrm{div}_x(\mathbf{U} - \mathbf{u})\,(F_c(c) - F_c(C))\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau\int_\Omega \big[\Delta_x(C - c)(F_c'(c) - F_c'(C))\big]\,\mathrm{d}x\,\mathrm{d}t + c_4\int_0^\tau \mathcal{E}\big(\rho, c, \mathbf{u} \,\big|\, r, C, \mathbf{U}\big)\,\mathrm{d}t.
\end{aligned}
$$

Using the boundedness of c and C, we get:

$$\operatorname{div}_x(\mathbf{U} - \mathbf{u})(F_c(c) - F_c(C)) + \Delta_x(C - c)(F_c'(c) - F_c'(C)) \lesssim |\nabla_x(\mathbf{u} - \mathbf{U})| + |\Delta_x c - \Delta_x C|\,|c - C|.$$

Using the Korn-Poincaré inequality, we obtain:

$$\Big[\mathcal{E}\big(\rho, c, \mathbf{u} \mid r, C, \mathbf{U}\big)\Big]_{t=0}^{t=\tau} \lesssim \int_0^\tau \mathcal{E}\big(\rho, c, \mathbf{u} \mid r, C\big)$$

Application: incompressible limit

Existence of the limit problem (incompressible NSAC):

Lin, Liu (1995): Existence of global weak solutions, existence of strong solution globally in 2D and locally in 3D

Thanks to the energy inequality and the continuity equation, $[\rho_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon]$ satisfy:

$$\int_\Omega \frac{1}{2}\rho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{2}|\nabla_x c_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left(F_e(\rho_\varepsilon) - F_e'(1)(\rho_\varepsilon - 1) - F_e(1)\right)$$

We can write

$$\frac{1}{\varepsilon^2}\left(F_e(\rho_\varepsilon) - F_e'(1)(\rho_\varepsilon - 1) - F_e(1)\right) \gtrsim \left[\frac{\rho_\varepsilon - 1}{\varepsilon}\right]^2_{\mathrm{ess}} + \frac{(1+\rho_\varepsilon)_{\mathrm{res}}}{\varepsilon^2}. \qquad (15)$$

Using the boundedness of the energy

$\rho_\varepsilon(t$

$$\mathcal{E}_\varepsilon\left(\rho, \mathbf{u}, c \mid \mathbf{U}, C\right) = \int_\Omega \frac{1}{2}\rho|\mathbf{u} - \mathbf{U}|^2 + \frac{1}{2}|\nabla_x c - \nabla_x C|^2$$

$$+ \frac{1}{\varepsilon^2}\left(F_e(\rho) - F_e'(1)(\rho - 1) - F_e(1)\right) \, dx$$

$$\left[\mathcal{E}_\varepsilon\left(\rho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \mid C, \mathbf{U}\right)\right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \Big[\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : (\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}) + \mu_\varepsilon^2$$

Using the same treatment for the convective terms, we get:

$$
\begin{aligned}
&\mathcal{E}_\varepsilon\left(\rho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \,\middle|\, C, \mathbf{U}\right)\Big]_{t=0}^{t=\tau} + \frac{1}{2}\int_0^\tau\int_\Omega \Big[(\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u}_\varepsilon - \nabla_x \mathbf{U}) + \mu_\varepsilon^2 \Big] \,\mathrm{d}x\,\mathrm{d}t \\
&\leq -\int_0^\tau\int_\Omega [F_c(c_\varepsilon)\mathrm{div}_x \mathbf{u}_\varepsilon + \mu_\varepsilon F_c'(c_\varepsilon)] \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau\int_\Omega (\mathbf{U} - \mathbf{u}_\varepsilon)\cdot(-\nabla_x \Pi - \mathrm{div}_x(\nabla_x C \otimes \nabla_x C)) \,\mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau\int_\Omega (\nabla_x c_\varepsilon \otimes \nabla_x c_\varepsilon) : \nabla_x \mathbf{U} \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau\int_\Omega \partial_t(\Delta_x C) c_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_\Omega \Delta_x C\,[\mu_\varepsilon - \mathbf{u}_\varepsilon\cdot\nabla_x c_\varepsilon] \,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^\tau\int_\Omega \partial_t \frac{1}{2}|\nabla_x C|^2 \,\mathrm{d}x\,\mathrm{d}t + c_1\int_0^\tau \mathcal{E}_\varepsilon\left(\rho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \,\middle|\, C, \mathbf{U}\right)\mathrm{d}t.
\end{aligned} \tag{18}
$$

The only term different to the weak-strong uniqueness case is the one in $\nabla_x \Pi - \nabla_x F_c(C)$:

$$\Big[\mathcal{E}_\varepsilon\Big(\rho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \Big| C, \mathbf{U}\Big)\Big]_{t=0}^{t=\tau} \leq \int_0^\tau \mathcal{E}_\varepsilon\Big(\rho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \Big| C, \mathbf{U}\Big)\, dt + \int_0^\tau \int_\Omega (\mathbf{U} - \mathbf{u}_\varepsilon) \cdot \nabla_x(\Pi - F_c(C))\, \mathrm{d}x\, \mathrm{d}t.$$

From the assumptions on the initial conditions $\mathcal{E}_\varepsilon\Big(\rho_\varepsilon, c_\varepsilon, \mathbf{u}_\varepsilon \Big| C, \mathbf{U}\Big)(0) \to 0$ as $\varepsilon \to 0$.

Using the continuity equation we also have:

$$\int_0^\tau \int_\Omega (\mathbf{U} - \mathbf{u}_\varepsilon) \cdot \nabla_x \bar{\Pi}\, \mathrm{d}x\, \mathrm{d}t = -\int_0^\tau \int_\Omega (1 - \rho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x \bar{\Pi}\, \mathrm{d}x\, \mathrm{d}t - \Big[\int_\Omega \rho_\varepsilon \bar{\Pi}\, \mathrm{d}x\Big]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \rho_\varepsilon \partial_t \bar{\Pi}\, \mathrm{d}x\, \mathrm{d}t,$$

with $\bar{\Pi} = \Pi - F_c(C)$ and this term converges uniformly to 0 over $\tau \in [0, T]$.

Application: numerical approximation

Work in collaboration with Eduard Feireisl and Bangwei She:

1. Discretise the problem by a stable and consistent scheme
2. Prove that the numerical solutions converge to a dissipative weak solution and thus, by the weak-strong uniqueness principle, prove the unconditional convergence to a strong solution if the later exists

Starting point:
T. Karper: A convergent FEM-DG method for compressible Navier-Stokes system, 2013 ($\gamma > 3$)
E. Feireisl and M. Lukáčová-Medvid'ová: Convergence of a mixed finite element–discontinuous Galerkin scheme for the isentropic Navier-Stokes system via dissipative measure-valued solutions, 2018 ($\gamma > 1$)
Here and hereafter we suppose that

$$p \in C[0,\infty) \cap C^2(0,\infty),\ p(0) = 0,\ p'(\rho) > 0 \text{ for } \rho > 0;$$

the pressure potential

Mesh

Let $\mathcal{T}$

Function spaces

$\mathcal{P}^{`}_d(K$

Discrete Laplace operator

For any $v \in X_h$ we define $\Delta_h v \in W_h := \{v \in X_h | \int_\Omega v \, \mathrm{d}x = 0\}$ such that

$$-\int_\Omega \Delta_h v \, w \, \mathrm{d}x = B(v, w) \quad \text{for any } w \in W_h, \tag{21}$$

$$B(v, w) = \int_\Omega \nabla_h v \cdot \nabla_h w \, \mathrm{d}x + \sum_{\sigma \in \mathcal{E}} \int_\sigma \left([[w]]\mathbf{n} \cdot \{\!\{ \nabla_h v \}\!\} + [[v]]\mathbf{n} \cdot \{\!\{ \nabla_h w \}\!\} + \frac{c_B}{h^{1+\beta}} [[v]][[w]] \right) \mathrm{d}\sigma,$$

where $c_B > 0$ is a sufficient large constant to ensure the coercivity. The following identity holds:

$$B(v, v - w) = \frac{1}{2} B(v, v) - \frac{1}{2} B(w, w) + \frac{1}{2} B(v - w, v - w).$$

Furthermore, we define the following seminorms

$|v$

[redacted]

Numerical scheme

$$\int_\Omega D_t \rho_h^k \phi_h \, dx - \sum_{\sigma \in \mathcal{E}} \int_\sigma F_h^{\rm up}(\rho_h^k, \mathbf{u}_h^k)[[\phi_h]] \, d\sigma = 0, \quad \forall \phi_h \in Q_h;$$

$$\int_\Omega D_t(\rho_h^k \mathbf{u}_h^k) \cdot \varphi_h \, dx - \sum_{\sigma \in \mathcal{E}} \int_\sigma \mathbf{F}_h^{\rm up}(\rho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) \cdot [[\varphi_h]] \, d\sigma + \nu \int_\Omega \nabla_h \mathbf{u}_h^k : \nabla_h \varphi_h \, dx$$

$$+ \eta \int_\Omega {\rm div}_h \mathbf{u}_h^k {\rm div}_h \varphi_h \, dx = \int_\Omega p_h^k {\rm div}_h \varphi_h \, dx + \int_\Omega (f_h^k - \Delta_h c_h^k) \nabla_h c_h^k \cdot \varphi_h \, dx, \; \forall \varphi_h \in \mathbf{V_h};$$

$$\int_\Omega (D_t c_h^k + \mathbf{u}_h^k \cdot \nabla_h c_h^k) \psi_h \, dx = \int_\Omega \left(\Delta_h c_h^k - f_h^k \right) \psi_h \, dx, \quad \forall \psi_h \in X_h,$$

where $D_t v_h^k = \frac{v_h^k - v_h^{k-1}}{\Delta t}$ for all $k = 1, \dots, N_T$, $p_h^k = p(\rho_h^k)$, $\eta = \frac{d-2}{d}\nu + \lambda > 0$ and

$$f_h^k = \begin{cases} 2(c_h^k + 1) & \text{if } c_h^k \in (-\infty, -1), \\ (c_h^k)^3 - c_h^{k-1} + 1 & \dots \end{cases}$$

Some estimates

Lemma (Sobolev inequality)

Let $r \geq 0$ be a function defined on $\Omega \subset \mathbb{R}^d$ such that

$$0 < c_M \leq \int_\Omega r \,\mathrm{d}x, \;\textit{and}\; \int_\Omega r^\gamma \,\mathrm{d}x \leq c_E \;\textit{for}\; \gamma > 1,$$

where c_M and c_E are some positive constants. Then

$$\|v_h\|^2_{L^q(\Omega)} \lesssim c(\|r\,{}_h v_h\|^2_{L^2(\Omega)} + \int_\Omega r|\nabla_h v_h|^2 \,\mathrm{d}x)$$

for any $v_h \in \mathbf{V_h}$, and $1 \leq q \leq 6$ for $d = 3$, $1 \leq q < \infty$ for $d = 2$, where $c = c(c_M, c_E)$.

Stability

Conservation of mass:

$$\int_\Omega \rho_h^k \, dx = \int_\Omega \rho_h^{k-1} \, dx = \cdots = \int_\Omega \rho_h^0 \, dx = \int_\Omega \Pi_h^Q \rho_0 \, dx = \int_\Omega \rho_0 \, dx, \ \forall k = 1, \dots, N_T.$$

Discrete energy balance: Let $(\rho_h^k, \mathbf{u}_h^k, c_h^k)$ satisfy th [redacted]

$$D_t \int_\Omega \left(\frac{1}{2} \rho_h^k \left| \hat{\mathbf{u}}_h^k \right|^2 + P(\rho_h^k) \right) dx + D_t \int_\Omega F(c$$ [redacted]

$$+ \nu \| \nabla_h \mathbf{u}_h^k \|_{L^2}^2 + \eta \| \mathrm{div}_h \mathbf{u}_h^k \|_{L^2}^2 + \| D_t c_h^k + \mathbf{u}$$ [redacted]

where $D_{num} > 0$ is the numerical dissipation

$$D_{num} = \frac{\Delta t}{2} \int_\Omega \rho_h^{k-1} | D_t \hat{\mathbf{u}}_h^k |^2 \, dx + \frac{1}{2} \sum$$

There exist $\xi \in \mathrm{co}\{\rho_h^{k-1}, \rho_h^k\}$ and $\zeta \in \mathrm{co}\{\rho_K^k, \rho_L^k\}$ for any $\sigma = K|L \in \mathcal{E}$ such that

$$\int_\Omega D_t P(\rho_h^k)\,\mathrm{d}x + \int_\Omega p(\rho_h^k)\mathrm{div}_h \mathbf{u}_h^k\,\mathrm{d}x$$
$$= -\frac{\Delta t}{2}\int_\Omega P''(\xi)|D_t \rho_h^k|^2\,\mathrm{d}x - \sum_{\sigma\in\mathcal{E}}\int_\sigma P''(\zeta)[[\rho_h^k]]^2 \left(h^\varepsilon + \frac{1}{2}|\mathbf{u}_\sigma^k \cdot \mathbf{n}|\right)\mathrm{d}\sigma \le 0.$$

Take $\phi_h = D_t c_h^k + \mathbf{u}_h^k \cdot \nabla_h c_h^k \in X_h$:

$$\int_\Omega \left(D_t c_h^k + \mathbf{u}_h^k \right.$$

Uniform bounds

Lemma

Let $(\rho_h, \mathbf{u}_h, c_h)$ *be a solution to the numerical scheme for* $\gamma > 1$*. Then:*

$$k\rho_h|\mathbf{u}_h|^2k_{L^1L^1} \ . \ 1, \quad k\rho_h k_{L^1L^\gamma} \ . \ 1, \quad k\rho_h\mathbf{u}_h k_{L^1L^{\frac{2\gamma}{\gamma+1}}} \ . \ 1,$$

$$k\, r_h\mathbf{u}_h k_{L^2L^2} \ . \ 1, \quad k\mathrm{div}_h\mathbf{u}_h k_{L^2L^2} \ . \ 1, \quad k\mathbf{u}_h k_{L^2L^p} \ . \ 1, \quad k\rho_h\mathbf{u}_h k_{L^2L^q} \ . \ 1,$$

$$\sup_{t2(0,T)} |kc_h k| \approx \sup_{t2(0,T)} |kc_h(t)k|_B \ . \ 1, \quad kf_h k_{L^1L^2} \approx kc_h k_{L^1L^2} \ . \ kF(c_h)k_{L^1L^1} \ . \ 1,$$

$$kc_h k_{L^1L^p} \ . \ \sup_{t2(0,T)} |kc_h k|_B + kc_h k_{L^1L^2} \ . \ 1, \quad kD_t c_h + \mathbf{u}_h \cdot r_h c_h k_{L^2L^2} \ . \ 1,$$

$$k\Delta_h c_h k_{L^2L^2} \ . \ 1, \quad kD_t c_h k_{L^2L^{3=2}} \ . \ 1.$$

where $p\ 2\ [1,\ 1\ )$, $q\ 2\ [1, \gamma)$ *if* $d = 2$ *or* $p = 6$, $q = \frac{6\gamma}{\gamma+6}$ *if* $d = 3$.

The idea of the proof is to construct a mapping $\mathcal{F}$ that satisfies the topological degree theory. Define:

$$V = \quad (\rho_h^k, \mathbf{U}_h^k) \; 2 \; Q_h \times \mathcal{M}_h, \; \rho_h^k > 0 \quad ,$$

$$W = \quad (\rho_h^k, \mathbf{U}_h^k) \; 2 \; Q_h \times \mathcal{M}_h, \; k\mathbf{U}_h^k k \; 6 \; C_2, \epsilon < \rho_h^k < C_1 \quad ,$$

where $\mathbf{U}_h^k := (\mathbf{u}_h^k, c_h^k) \; 2 \; \mathbf{V_h} \times X_h =: \mathcal{M}_h$ and the norm $kU_h^k k$ is given by $kU_h^k k \equiv k\mathbf{u}_h^k k_{L^6} + kc_h^k k_{L^2}$.

h, ρ^k_h, ∂^k F/F104 5.9776 Tf 6.034 58300

Next, for ζ 2 $[0,1]$ and $U^? = (\mathbf{u}^?, c^?)$ we define

$$F : V \times [0,1] \; ! \; Q_h \times M_h, \quad (\rho_h^k, \mathbf{U}_h^k, \zeta) \quad ,$$

Consistency

Theorem

Let (ρ