Magnetorheological uids

Figure: Magnetite particles aggregating into chains. Image from K. Jiangang et al (Miner. Enginrg. '15)

- Suspension of non-colloidal ferromagnetic particles in a non-magnetizable uid
  - ! Brownian motion e ects are neglected
- .05-10 m size particles
  - ! Volume fractions of 10% to 50%
- Once a magnetic eld is applied, the particles organize in chain structures
- Millisecond transformation form uid to semi-solid state

Typical modeling approaches

Phenomenological approach

- Jacob Rabinow (AIEE Trans., '48)
- Basic mathematical model by Rosensweig & Neuringer (Phys. Fluids, '64)
  - Shliomis (Sov. Phys. JETP, '72) improves model by allowing \internal rotations"
- Classical thermodynamics approach
  -

Cauchystress

- Magnetorheological uids exhibit non-Newtonian behavior
- In shear experiments the Bingham constitutive law models response of magnetotheological uids

Newtonian incompressible uids

$$= \quad p\,I + 2 \quad e(\mathbf{v}); \quad e(\mathbf{v}) = \frac{1}{2}(r\ \mathbf{v} + r\ ^{t}\mathbf{v})$$

$$^{0} = 2 \quad e^{0}(\mathbf{v}); \quad A^{0}(\mathbf{v}) = A \quad \frac{1}{n}\,\mathrm{tr}\,(A)$$

Bingham incompressible uids

$$\begin{cases} \text{if } j \quad j \qquad _{y};\ \text{then} \quad = 2 \quad e(\mathbf{v}) + \quad _{y}\frac{e(\mathbf{v})}{je(\mathbf{v})j} \\ \text{if } j \quad j \qquad _{y};\ \text{then } e(\mathbf{v}) = 0 \end{cases}$$

Figure: stress versus strain rate

Balance of forces and torques

The force can be written in terms of the magnetic Maxwell stress,

$$\mathbf{F} := \frac{1}{2}|\mathbf{H}|^2\, r \quad () \quad \mathbf{F} = \operatorname{div}(\ {}^{\text{mag}}) \quad \mathbf{B} \quad \operatorname{curl}(\hat{}\,\mathbf{B});$$

$$ {}^{\text{mag}} := \hat{}\,\mathbf{B} \quad \mathbf{B} \quad \frac{1}{2}\hat{}\,|\mathbf{B}|^2 I \quad =) \quad \operatorname{div}(\ {}^{\text{mag}}) = \begin{cases} 0 & \text{if } \mathbf{x} \ 2 \quad {}_{F}; \\ \mathbf{B} \quad \operatorname{curl}(\hat{}\,\mathbf{B}) & \text{if } \mathbf{x} \ 2 \quad {}_{P}: \end{cases}$$

Hence, we can write the balance of forces and torques on each particle as

$$0 = \int_{@P(\ )} \mathbf{n}^{(\ )}\, ds + \int_{@P(\ )} J\ {}^{\text{mag}}\mathbf{n}^{(\ )} K ds \quad \int_{P(\ )} \mathbf{B} \quad \operatorname{curl}(\hat{}\,\mathbf{B})\, dx;$$

$$0 = \int_{@P(\ )} \mathbf{n}^{(\ )} \quad \mathbf{x} \quad \mathbf{x}^{(\ )}\, ds + \quad \int_{@P(\ )} J\ {}^{\text{mag}}\mathbf{n}^{(\ )} K \quad \mathbf{x} \quad \mathbf{x}^{(\ )}\, ds$$

$$\quad \int_{P(\ )} (\mathbf{B} \quad \operatorname{curl}(\hat{}\,\mathbf{B})) \quad \mathbf{x} \quad \mathbf{x}^{(\ )}\, dx:$$

$\quad = \ \overline{\ }\,\overline{H}\,\overline{L} = \overline{\ }\,\overline{V}$ is the Alfven number

Some results regarding function spaces

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Let $\mathcal{O} \subset \mathbb{R}^d$ be any open, bounded, multiply connected set with boundary $\Gamma := \partial\mathcal{O}$ of class C^2. The exterior boundary will be denoted by Γ_0 and by Γ_j; $j = 1, \dots, N-1$, the other components of Γ. Define Y to be the Hilbert space of vector fields,

$$Y := \left\{ \mathbf{u} \in L^2(\mathcal{O};\mathbb{R}^d) \mid \operatorname{div}\mathbf{u} \in L^2(\mathcal{O});\ \operatorname{curl}(\hat{\mathbf{u}}) \in L^2(\mathcal{O};\mathbb{R}^d);\ \mathbf{u}\cdot\mathbf{n} \in H^{1/2}(\Gamma_0) \right\};$$

for the norm,

$$\|\mathbf{w}\|_Y := \|\mathbf{w}\|_{L^2(\mathcal{O};\mathbb{R}^d)} + \|\operatorname{div}\mathbf{w}\|_{L^2(\mathcal{O})} + \left\|\operatorname{curl}(\hat{\mathbf{w}})\right\|_{L^2(\mathcal{O};\mathbb{R}^d)} + \|\mathbf{w}\cdot\mathbf{n}\|_{H^{1/2}(\Gamma_0)};$$

then for all $\mathbf{w} \in Y$ we have, $\mathbf{w}|_{\mathcal{O}_i} \in H^1(\mathcal{O}_i;\mathbb{R}^d)$ for $i = 1, \dots, N$ and

$$\left\|\mathbf{w}|_{\mathcal{O}_i}\right\|_{H^1(\mathcal{O}_i;\mathbb{R}^d)} \le C_{\mathcal{O}_i}\|\mathbf{w}\|_Y :$$

(small) extension of Prop. 3.1 in Foias & Temam (Ann. Sc. norm. super. Pisa, '78)

Some results regarding function spaces

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Define a new norm on Y by

$$[\mathbf{w}]_Y := \|\operatorname{div}\mathbf{w}\|_{L^2(O)} + \|\operatorname{curl}(\wedge \mathbf{w})\|_{L^2(O;\mathbb{R}^d)} + \|\mathbf{w}\cdot\mathbf{n}\|_{H^{1/2}(\Gamma_0)};$$

then Y is also a Hilbert space with norm $[\cdot]_Y$.

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There exists a constant, $c := c(O)$, such that

Inner product space for the velocity,

$$V = \left\{ \mathbf{v} \in H^1_0(\Omega_F; \mathbb{R}^d) \mid \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_F;\ \mathbf{v} = \mathbf{v}^{(\ell)} + \boldsymbol{\omega}^{(\ell)} \times (\mathbf{x} - \mathbf{x}^{(\ell)}) \text{ on } \partial P^{(\ell)} \right\}.$$

$$(\mathbf{v} \mid \boldsymbol{\varphi})_V = \int_{\Omega_F} 2\, e(\mathbf{v}) : e(\boldsymbol{\varphi})\, d\mathbf{x}.$$

Inner product space for the magnetic induction,

$$Y = \left\{ \mathbf{w} \in L^2(\Omega; \mathbb{R}^d) \mid \operatorname{div}(\mathbf{w}) \in L^2(\Omega);\ \operatorname{curl}(\hat{\ }\mathbf{w}) \in L^2(\Omega; \mathbb{R}^d);\ \mathbf{w} \cdot \mathbf{n} \in H^{1/2}(\Gamma_0) \right\};$$

$$(\mathbf{h} \mid \boldsymbol{\psi})_Y = \int_\Omega \operatorname{div}(\mathbf{h}) \operatorname{div}(\boldsymbol{\psi})\, d\mathbf{x} + \int_\Omega \operatorname{curl}(\hat{\ }\mathbf{h}) \cdot \operatorname{curl}(\hat{\ }\boldsymbol{\psi})\, d\mathbf{x} + \int_{\Gamma_0} (\mathbf{h} \cdot \mathbf{n})(\boldsymbol{\psi} \cdot \mathbf{n})\, ds.$$

Variational formulation of Stokes' equation

Multiply with am appropriate test function,

$$\int_{\cup_{=1}^{K} \partial P(\ )} \mathbf{n}(\ ) \cdot \ ds + \int_{F} 2e(\mathbf{v}) : e(\ )\, d\mathbf{x} = 0.$$

Use balance of forces and torques,

$$\int_{\cup_{=1}^{K} \partial P(\ )} J \ {}^{mag}\mathbf{n}(\ ) K \cdot \ ds \quad \int_{P} \mathbf{B} \quad \mathrm{curl}(\hat{\ }\mathbf{B}) \cdot \ d\mathbf{x}$$

$$+ \int_{F} 2\, e(\mathbf{v}) : e(\ )\, d\mathbf{x} = 0.$$

Find $\mathbf{v} \in V$,

$$(\mathbf{v} | \ )_V + \int_{F} {}^{mag} : e(\ )\, d\mathbf{x} = \mathbf{0} \quad \text{for all} \quad \in V.$$

Augmented variational formulation of Maxwell's equations

For an appropriate test function,

- multiply the divergence part by $\frac{}{R_m}$ div( )
- multiply the rotational part by $\frac{}{R_m}$ curl(^)
- multiply the exterior boundary condition by $\frac{}{R_m}$ **n**

$$(\mathbf{h} \mid \ )_Y = \int \mathrm{div}(\mathbf{h})\,\mathrm{div}(\ )\,d\mathbf{x} + \int \mathrm{curl}(\wedge\,\mathbf{h})\,\mathrm{curl}(\wedge\ )\,d\mathbf{x} + \int_0 (\mathbf{h}\ \mathbf{n})(\ \mathbf{n})\,ds.$$

Find $\mathbf{B} \in Y$ such that,

$$\frac{}{R_m}(\mathbf{B} \mid \ )_Y = \int_P [\mathbf{v}\ \ \mathbf{B}]\,\mathrm{curl}(\wedge\ )\,d\mathbf{x} + \frac{}{R_m}\int_0 (\mathbf{b}^0\ \mathbf{n})(\ \mathbf{n})\,ds;$$

for all $\in Y$.

Variational formulation of the problem

Find $(\mathbf{v}; \mathbf{B}) \; 2 \; V \;\; Y$ such that,

$$(\mathbf{v} \; j \;\;)_V + \frac{}{R_m} (\mathbf{B} \; j \;\;)_Y = \int_F {}^{mag} : e(\;) \, d\mathbf{x} + \int_P [\mathbf{v} \;\; \mathbf{B}].\mathrm{curl}(\hat{} \;\;) \, d\mathbf{x}$$

$$+ \frac{}{R_m} \int_0 (\mathbf{b}^0 . \mathbf{n})(\;\; . \mathbf{n}) \, ds;$$

for all $(\; ; \;\;) \; 2 \; V \;\; Y$. Naturally, a norm is associated to the above inner (cross-) product space denoted by $jjj (\; ; \;) jjj := k \; k_V + \frac{}{R_m} [\;]_Y$.

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The pair $(\mathbf{v}; \mathbf{B})$ satis es the strong form of Maxwell's and Stokes' equations as well as their BC if and only if it is a solution to the above weak formulation.

Equivalence between weak and strong form of the problem

One direction is clear

Recover Maxwell's equations: introduce : $\mathbb{R}^d$! [0; 1]

$$(\mathbf{x}) = \begin{cases} 1 & \text{if } d(\mathbf{x}; \ _0) < \ ; \\ 0 & \text{if } d(\mathbf{x}; \ _0) > 2 \end{cases}$$

Using the approach of Ledyzhenskaya$_{(a3)}$, de ne $(\mathbf{x}) := (b_2^0 x_3; b_3^0 x_1; b_1^0 x_2)$.
Set $\mathbf{a}$ ($\mathbf{x}$

The test function

Construct $\mathbf{W} := \ \mathbf{V} \quad r\ \ p$ and verify,

$$\mathrm{div}(\mathbf{W}) = 0; \quad \mathrm{curl}(\wedge\mathbf{W}) = R_m\mathbf{v} \quad \mathbf{B} \quad {}_P; \quad \mathbf{W}.\mathbf{n} = 0$$

Construct $\ := \mathbf{B} \quad \mathbf{W} \quad \mathbf{a} \ \ 2\ Y$,

$$\int \mathrm{div}\,\mathbf{B}\,\mathrm{div}(\mathbf{B} \quad \mathbf{W} \quad \mathbf{a})\,dx$$
$$+ \int [\mathrm{curl}(\wedge\mathbf{B}) \quad R_m\mathbf{v} \quad \mathbf{B} \quad {}_P].\mathrm{curl}(\wedge(\mathbf{B} \quad \mathbf{W} \quad \mathbf{a}))\,dx$$
$$+ \int_0 [(\mathbf{B} \quad \mathbf{b}^0).\mathbf{n}][(\mathbf{B} \quad \mathbf{W} \quad \mathbf{a}).\mathbf{n}]\,ds = 0$$

Using the properties of the vector elds $\mathbf{W}$ and $\mathbf{a}$ we obtain:

$$\int |\mathrm{div}\mathbf{B}|^2\,dx + \int |\mathrm{curl}(\wedge\mathbf{B}) \quad R_m\mathbf{v} \quad \mathbf{B} \quad {}_P|^2\,dx + \int_0 (\mathbf{B} \quad \mathbf{b}^0).\mathbf{n}^{\,2}\,ds = 0$$

The Altman-Shinbrot fxed point theorem

Let H denote a real or complex Hilbert space, and S_r and B_r denote the sphere and the closed ball of radius r centered at zero, respectively:

$$S_r = \{x \in H \mid \|x\|_H = r\}; \quad B_r = \{x \in H \mid \|x\|_H \leq r\}.$$

Theorem (Altman, Bull. Acad. Polon. Sci. '57; Shinbrot, ARMA '64)

Let $\mathcal{H}$ be an operator on the separable Hilbert space H, continuous in the weak topology on H. If there is a positive constant r such that $\Re(\mathcal{H}x, x) \leq \|x\|_H^2$ for all $x \in B_r$, then $\mathcal{H}$ has a fxed point in B_r.

Corollary: Let G be an operator on the separable Hilbert space H, continuous in the weak topology on H. Let y be an element of H. If there exists a positive r such that either $\Re(Gx - y, x) \geq 0$ for all $x \in S_r$ OR $\Re(Gx - y, x) \leq 0$ for all $x \in S_r$ then y is in the range of G.

Corollary: Let G be an operator on the separable Hilbert space H, continuous in the weak topology on H. Then, zero is in the range of G if (Gx, x) is of one sign on some sphere S_r.

Existence

For all $\mathbf{v}; \mathbf{B}; \boldsymbol{\varphi}$ we defne the following expression Q by,

$Q[(\mathbf{v}; \mathbf{B}); (\boldsymbol{\varphi} \quad)] :=$; 9 9.9626 Tf 7.324 0 Td [(5.5519]TJ/F33 Fedeartz and Riesz's fol32(allem)]TJ/F33 s

Existence

Lemma

The nonlinear operator $F : (\mathbf{v};\mathbf{B}) \mapsto F(\mathbf{v};\mathbf{B})$ is continuous in the weak topology of the product space $V \times Y$.

Lemma

If the magnetic Reynolds number, R_m, is small then

$$(F(\mathbf{v};\mathbf{B});(\mathbf{v};\mathbf{B})) \geq \frac{1}{2}|||(\mathbf{v};\mathbf{B})|||^2 \text{ for all } (\mathbf{v};\mathbf{B}) \in V \times Y:$$

Theorem

If the magnetic Reynolds number, R_m satisfies R_m, F80

$$1 \quad 2^{C_{FP}|\mathbf{b}^0| mes_d}$$

Existence and comments

Apply Altman-Shinbrot theorem to the operator equation

Show there exists r such that

$$(F(\mathbf{v};\mathbf{B}) \quad (\mathbf{f};\mathbf{g});(\mathbf{v};\mathbf{B})) \geq 0$$

for all $(\mathbf{v};\mathbf{B})$ with $jjj(\mathbf{v};\mathbf{B})jjj = r$

Select $r = 2\ jjj(\mathbf{f};\mathbf{g})jjj$

The case of $R_m \quad 0$ can be thought of as a limit case of the above model.

$R_m \quad 0$ the system becomes weakly coupled and, existence and uniquenes follow by invoking the Lax-Milgram lemma, once higher integrability of the magnetic induction is established

In two spatial dimensions system can also be solved analytically. Resulting behavior is of a Bingham type fuid.